Brief Overview of the Petz recovery map and its applications

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The Petz recovery map is a prevalent tool in various fields working with quantum systems. This brief review paper will provide a conceptual understanding of the Petz recovery map from a classical information perspective. Moreover, the paper will discuss the methods in current advances within the field to solve for a Petz recovery map exactly by looking into Gaussian channels, and approximate results through an algorithmic method.

INTRODUCTION

The Petz transpose map has been a ubiquitous tool in quantum information theory and has been at the forefront of research within this field. Originally discovered by D. Petz in the 1980s [1], it was further rediscovered within a different context in quantum error correction [2] and within quantum statistical mechanics [3].

The Petz recovery channel can be thought of as a quantum analog of Bayes theorem; it is the idea that there exists a channel $\mathcal{P}^{\sigma,\mathcal{E}}_{y\to x}$ that can completely reverse the action of the quantum channel $\mathcal{E}_{x\to y}(\rho)$. More on this will be discussed later in the paper.

This paper will be a brief literature review on the recent challenges in determining the method to obtain the recovery channel $\mathcal{P}^{\sigma,\mathcal{E}}_{y\to x}$ is. There is a wide array of topics of interest within this subject from its use in quantum thermodynamics [4], estimating approximate reversibility [5] and applications quantum statistical mechanics [6] [7] and to quantum gravity [8]. This paper looks into its applications in Fermionic Gaussian channels and a quantum algorithm to obtain Petz recovery channels.

Hence, Section I is a brief mathematical introduction to Petz map that will give readers a better conceptual understanding of the material. Section II will discuss the advances to solve the Petz recovery channel exactly for Fermionic Gaussian Channels, and Section III will discuss an approximate algorithm for the Petz recovery channel.

I. CONSTRUCTION OF THE PETZ RECOVERY MAP

Recall in the classical "channel" case, the Bayes rule determines the channel dynamics; which is to say that $p(y) = \sum_{x} p(x)p(y|x)$. Here, the quantum analog of the classical channel p(y|x) is $\mathcal{E}_{x \to y}$, and similarly, σ_x , ρ_y are the quantum analog of the input state p(x), and output state p(y) respectively. The analogies above can be extended to the channel p(x|y) that has a quantum analog that is $\mathcal{P}^{\sigma,\mathcal{E}}_{y\to x}$, which is the Petz recovery channel. Here, the superscripts are there for clarity, indicating that it is a function of the input states σ_x , and the quantum constraints are the superscripts.

tum channel $\mathcal{E}_{x \to y}$. The subscript indicates that this is a channel to describe the reverse process of taking the output ρ_y to the input σ_x . The construction of the Petz recovery channel is as follows:

$$\mathcal{P}^{\sigma,\mathcal{E}}{}_{y \to x}(\bullet) = \sigma_x^{\frac{1}{2}} \mathcal{E}^{\dagger} \left(\mathcal{E}(\sigma_x)^{-\frac{1}{2}}(\bullet) \mathcal{E}(\sigma_x)^{-\frac{1}{2}} \right) \sigma_x^{\frac{1}{2}}$$
(1)

This is comprised basically of three operations

1.
$$\star \equiv \mathcal{E}(\sigma_x)^{-\frac{1}{2}}(\bullet)\mathcal{E}(\sigma_x)^{-\frac{1}{2}}$$
(2)

$$2. \quad * \equiv \mathcal{E}^{\dagger}(\star) \tag{3}$$

3.
$$\mathcal{P}^{\sigma,\mathcal{E}}_{y\to x}(\bullet) = \sigma_x^{\frac{1}{2}} * \sigma_x^{\frac{1}{2}}$$
 (4)

where here, \mathcal{E}^{\dagger} is the adjoint channel. It is important to note that the Petz recovery channel is CPTP (Completely Positive and Trace Preserving), since all the operations in (2)-(4) are CP, and although each operation in (2)-(4) is not individually TP, the overall operation by putting them together, is TP. Recall that in Bayes theorem:

$$p(x|y) = \frac{p(x)p(y|x)}{p(y)}$$

we see the similarity between p(x|y) with $\mathcal{P}^{\sigma,\mathcal{E}}_{y\to x}(\bullet)$ as follows: $p(y) \to (2), p(x|y) \to (3)$ and $p(x) \to (4)$.

Hence, putting everything together, we can see that the recovery channel, acting on the original channel, gives back the original state. In other words:

$$\mathcal{P}^{\sigma,\mathcal{E}}_{y \to x} \circ \mathcal{E}_{x \to y}(\bullet) = \bullet$$

It is important to note here that although it is possible to understand the forward channel process in $\mathcal{E}_{x \to y}$, it does not mean that the reverse is available. Although there are some exact, and "simple" results, such as in situations containing unitary dynamics [9] and thermal operations [7], to name a few, there is also a handful of research in obtaining an approximate recovery channel, as we will see in the last couple section.

II. PETZ RECOVERY MAP FOR FERMIONIC GAUSSIAN CHANNEL [10]

This work constructed a Petz recovery map for Fermionic Gaussian channels using the Grassmann representation of the Fermionic Gaussian channels. Let's consider a Hamiltonian that is quadratic in its Fermionic creation and annihilation operators, given by;

$$H = \sum_{ij} c_i^{\dagger} K_{ij} c_j + c_i^{\dagger} A_{ij} c_j^{\dagger} + c_i A_{ij}^{\dagger} c_j \tag{5}$$

Here, c, c^{\dagger} denote the annihilation and creation operators that satisfy the anti-commutation relations, with K as a hermitian matrix, and A defined to be an anti-symmetric matrix, such that the Hamiltonian is hermitian. In this case, the thermal states of the Hamiltonian, are defined to be the Gaussian states, and are of the form;

$$\rho \equiv \frac{e^{-\beta H}}{Tr[e^{-\beta H}]} \tag{6}$$

In addition, recall that the Petz recovery map is defined with respect to a reference state σ . Let's define G^{ξ} as the covariance matrix of the state ξ . Here the covariance elements in the covariance matrix are defined to be;

$$C(x,y) \equiv \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})$$

The barred values denote the mean, and n is the sample size.

Here, looking back to the Hamiltonian given by (5), it is then possible to rewrite this in the Majorana Fermionic basis, which satisfies the algebra;

$$\gamma_{2i-1} = (c_i + c_i^{\dagger}), \quad \gamma_{2i} = i(c_i - c_i^{\dagger}); \quad \{\gamma_i, \gamma_j\} = 2\delta_{ij}$$

such that (5) transforms as:

$$H = \frac{i}{2} \sum_{ij}^{2n} \gamma_i M_{ij} \gamma_j \tag{7}$$

where M is a real anti-symmetric matrix defined by the Pauli matrices.

It is then possible to define the 2n Majorana operators by;

$$X = \alpha I + \sum_{k=1}^{2n} \sum_{1 \le a_1 \le \dots a_p \le n} \alpha_{a_1 a_2 \dots a_p} \gamma_{a_1} \gamma_{a_2} \dots \gamma_{a_p}$$

where $\alpha \equiv 2^{-n}TrX$. Instead of the Majorana basis, it is more favorable for the calculations to move to the Grassmann variables. Recall that the Grassmann variables are given by θ, η, \dots follows the anti-commutation relation $\theta_i \theta_j + \theta_j \theta_i = 0$, and subsequently, $\theta_i^2 = 0$. Since a general operator in the Grassmann variables takes the form

$$f = \alpha 1 + \sum_{k=1}^{2n} \sum_{1 \le a_1 \le \dots a_p \le n} \alpha_{a_1 a_2 \dots a_p} \theta_{a_1} \theta_{a_2} \dots \theta_{a_p}$$

we note then that there is an isomorphism between the states $\gamma_i \to \theta_i$, and $\mathbf{I} \to 1$. Hence, a Majorana operator can be mapped into a polynomial of Grassmann variables labeled by $\omega(X, \theta)$, abbreviated as $X(\theta)$.

Thus, the channel is Gaussian if it has the integral representation in the following form:

$$\mathcal{E}_{x \to y}(X, \theta) = C \int exp[S(\theta, \eta) + i\eta^T \mu] X(\mu) D\eta D\mu \quad (8)$$

where:

$$S(\theta,\eta) \equiv \frac{i}{2}(\theta^{T},\eta^{T}) \begin{pmatrix} A & B \\ B^{T} & D \end{pmatrix} \begin{pmatrix} \theta \\ \eta \end{pmatrix}$$
$$= \frac{i}{2}\theta^{T}A\theta + \frac{i}{2}\eta^{T}D\eta + i\theta^{T}B\eta \quad (9)$$

A, B, and D are complex 2x2 matrices (A & D are anti-symmetric matrices), and C is a complex number. However, the preserve the CPTP nature of the quantum channel, we are restricted to C = 1, D = 0. Moreover, defining the matrix above in (9) in red to be N, we must note that $N^T N \leq 1$.

Thus now, we know some of the general properties of the matrix A, B, and D should be like, for the Petz recovery channel.

To develop the exact values for these matrices, let's find an integral representation for equations (2) - (4). Here, We would like to use the properties that if \mathcal{E}_1 and \mathcal{E}_2 are two CPTP Gaussian maps, then $\mathcal{E}_1 \circ \mathcal{E}_2$ would also result in a CPTP Gaussian map as well. Using this property, and equation (8), it is possible to look for the integral representations of (2) - (4) as a function of Grassmann variables. Doing so, one can find the result as follows:

$$A_{1} = -G^{\mathcal{E}(\sigma)}, B_{1} = \sqrt{I_{2n} + (G^{\mathcal{E}(\sigma)})^{2}},$$

$$C_{1} = 2^{n} det(I_{2n} + (G^{\mathcal{E}(\sigma)})^{2})^{-\frac{1}{2}}, D_{1} = G^{\mathcal{E}(\sigma)} \quad (10)$$

$$A_2 = 0, B_2 = B^T, C_2 = 1, D_2 = -A$$
(11)

$$A_3 = G^{\sigma}, B_3 = \sqrt{I_{2n} + (G^{\sigma})^2},$$

 $C_1 = \frac{1}{2^n}, D_1 = -G^{\sigma}$ (12)

Here, the subscript under the Matrix components A, B, and D, and the complex number C indicates the steps given in equations (2)-(4). For example, A_1, B_1, D_1 , and

 C_1 are the complex matrix and number for the integral representation of the Gaussian map given in equation (2).

Hence, putting everything together, one can find that; the matrix components for the integral representation of the Petz recovery map for a Gaussian linear map are given by;

$$A_{\mathcal{P}} = G^{\sigma} - B_{\mathcal{P}}G^{\mathcal{E}(\sigma)}B_{\mathcal{P}}^{T}, \quad C_{\mathcal{P}} = 1, \quad D_{\mathcal{P}} = 0$$
$$B_{\mathcal{P}} = \sqrt{I_{2n} + (G^{\sigma})^{2}}B^{T} \left(\sqrt{I_{2n} + (G^{\mathcal{E}(\sigma)})^{2}}\right)^{-1} \quad (13)$$

Hence, with these complex matrices given in (13), we can show that a Petz recovery map can be constructed exactly for a Gaussian channel, in terms of the covariance matrix and the reference state σ . Similar work can be done with Bosonic systems. This is just a brief review. More complete derivations, with information regarding the fidelity (a measure of similarity between two states), and rotated Petz map for Gaussian channels, should refer to the by Swingle et al. [10].

III. QUANTUM ALGORITHM FOR PETZ RECOVERY CHANNEL [11]

Before proceeding further, it should be noted that this formalism delivered by this paper, was developed through the use of block encoding and the Quantum Singular Value Transformation (QVST) formalism [12].

Block encoding - To be able to apply the algorithm to a general transformation, the Block encoding method is used. Suppose we are interested in the transformation of $|\psi\rangle \rightarrow \propto A |\psi\rangle$, where A is an arbitrary complex matrix. If A happens to be non-unitary, non-square or has a large operator norm, then block encoding techniques can be utilized. Here we define a unitary matrix U as the block encoding of A; essentially, embed A within the unitary, and act the unitary to the state.

$$U = \begin{bmatrix} A/\alpha & \cdot \\ \cdot & \cdot \end{bmatrix} \longleftrightarrow A = \alpha(\langle 0 | \otimes I)U(|0\rangle \otimes I)$$

Where $\alpha \geq ||A||$. The idea is that first, on an n-qubit input state $|\psi\rangle$, $|\psi\rangle$ is enlarged by the tensor product with state $|0\rangle \otimes \psi$. The $|0\rangle$ states are known as Ancilla bits and are extra bits used for calculation purposes. In this case, this is so that the unitary U can act on the enlarged state and then we can measure the Ancilla bit. If the outcome is $|0\rangle$ (with probability $1/\alpha^2$), then we know that the n-qubit state $|\psi\rangle \rightarrow \propto A |\psi\rangle / \alpha$. This process is the probabilistic implementation of map A acting on n-qubits.

QVST - In conjunction with block-encoding, QSVT is a useful technique to provide the ability to do quantum matrix arithmetic. Take the block encoding of density operators ρ , given by U^{ρ} , which through QSVT, transforms $U^{\rho} \to U^{\tilde{f}(\rho)}$. Here $\tilde{f}(\rho)$ is a polynomial approximation of some function f, applied to the singular values of the density matrix ρ . Recall that the singular values s satisfy det(s $\mathbf{I} - \rho$) = 0. For example, if we look at equations (2) and (4), we see that the functions of interest are $f_1(\bullet) = \bullet^{-\frac{1}{2}}, f_2(\bullet) = \bullet^{\frac{1}{2}}$.

$$U^{\rho} = \begin{bmatrix} \rho & . \\ . & . \end{bmatrix} \xrightarrow{QSVT} U^{\tilde{f}(\rho)} = \begin{bmatrix} \tilde{f}(\rho) & . \\ . & . \end{bmatrix}$$

As a technique that relies on approximations, it is important to introduce the error δ , such that it can now be defined that U is an (a, δ) block encoding in A if $||A - \alpha(\langle 0| \otimes I)U(|0\rangle \otimes I|| \leq \delta$. QSVT allows the us to find an approximation $\tilde{f}_1(\bullet)$ such that:

$$\frac{1}{2}\left\|\tilde{f}_1(\bullet) - \bullet^{-\frac{1}{2}}\right\| \le \delta, \quad \frac{1}{2}\left\|\tilde{f}_2(\bullet) - \bullet^{\frac{1}{2}}\right\| \le \delta$$

We see from here that block encoding, in conjunction with QSVT, is used to solve equations (2) and (4) of the Petz recovery map. Equation (3) requires a way to rewrite the quantum channel adjoint as:

$$\mathcal{E}^{\dagger}(\bullet) = Tr_{\tilde{E}}[|0\rangle_{E'} (U_{E'X \to EY}^{\mathcal{E}})^{\dagger} (\Gamma_{E\tilde{E}} \otimes \bullet) U_{E'X \to EY}^{\mathcal{E}} |0\rangle_{E'}$$
(14)

Where here $\Gamma_{E\tilde{E}} \equiv |\Gamma\rangle \langle \Gamma|_{E\tilde{E}}$ is an operator denoting a maximally entangled state on E and reference system \tilde{E} , and that system E' is isomorphic to system E. Here, $|\Gamma\rangle_{E\tilde{E}} \equiv \frac{1}{d_E} \sum_{i}^{d_E} |i\rangle_E |i\rangle_{\tilde{E}}$. Thus, applying the techniques described above, one

Thus, applying the techniques described above, one can then construct an expression for the isometric extension of the Petz recovery channel, given by;

$$V_{Y \to \tilde{E}X}^{\mathcal{P}} \equiv (|0\rangle_{E'} \otimes I_{\tilde{E}} \otimes \sigma_X^{\frac{1}{2}}) (U_{E'X \to EY}^{\mathcal{E}})^{\dagger} (|\Gamma\rangle_{E\tilde{E}} \otimes [\mathcal{E}(\sigma_X)]^{-\frac{1}{2}})$$
(15)

Tracing over \tilde{E} in (15), recovers the Petz recovery channel $\mathcal{P}^{\sigma,\mathcal{E}}_{y\to x}$. Here, the algorithm is centered on equation (15), where each step of the algorithm performs a task within a part of (15).

The algorithm consists of 4 different steps.

- 1. Construct the unitary $U_{R'Y}^{f_1(\mathcal{E}(\sigma_X))}$, which is a block encoding of $[\mathcal{E}(\sigma_X)]^{-\frac{1}{2}}$ though QVST
- 2. Prepare the normalized maximally entangled state $|\Gamma\rangle_{E\tilde{E}}/\sqrt{d_E}$. Then apply the unitary $(U_{E'X\to EY}^{\mathcal{E}})^{\dagger}$ to the prepared state. This is similar to the probabilistic implementation of the quantum channel \mathcal{E} through U, but the measurement and post-selection step is done later.
- 3. In this step, we first construct a block-encoding of σ_X , by QVST, to $U_{R''X}^{f_2(\tilde{\sigma}_X)}$. Then apply the unitary $U_{R''X}^{f_2(\tilde{\sigma}_X)}$ to the output of step 2.

4. Lastly, performing the measurement on the R"ER' systems should provide a successful implementation of $V_{Y \to \tilde{E}X}^{\mathcal{P}}$, up to a normalization. Tracing over \tilde{E} in (15), recovers the Petz recovery channel $\mathcal{P}^{\sigma, \mathcal{E}}_{v \to x}$

Thus, this paper provided an algorithm to construct an approximate Petz recovery channel, up to some error. More information regarding the algorithm such as its error bounds, complexity, and applications to pretty good measurements, of which are out of the scope of this brief review, should refer to the results from the paper.

CONCLUSION

In conclusion, this brief literature review provides a conceptual understanding of the Petz recovery map and summarises some advances in the field. The Petz recovery map has been an important tool in various fields that are quantum related and will continue to be a developing field. Although current developments have provided more understanding, and have made its implementation more useful, as seen with the algorithm above, it is still far from complete.

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