Fault-Tolerant Quantum Computation

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Like in classical computation circuits, errors can occur in different stages of a quantum circuit and they can easily propagated. Therefore, a fault-tolerant circuit design must be introduced to contain the quantum error propagation. In this paper, we provide an overview based on Gottesman's paper [1] to various fault-tolerant quantum circuit gadgets and how they can possibly be used together to reduce the overall error rate of a quantum circuit.

I. INTRODUCTION

Error correction has been a popular area in classical information theory since 1940s. Hamming code [2], for example, is a well developed error correction system that has been widely used in computer memory. On the other hand, as quantum information theory and quantum computing develops, many also started to look at the error correction methods that can be potentially beneficial to quantum computations. Notable results include the nine-qubit code [3], stabilizer codes [1], quantum CSS codes (which is a special type of stabilizer code constructed from classical codes) [4] [5], etc.

A. Error in Quantum Circuits

Like the circuit components used in a classical computer, to process information in a quantum computer, we need to introduce some basic components in a quantum circuit [1]:

1. **Preparation.** Operations to prepare a (new) qubit in some standard state.

2. **Quantum Gates.** The gates H, CNOT, $R_{\pi/8}$ consist a sufficient group for quantum operations [6].

3. **Measurement.** Decoherent action to measure individual qubits in some standard basis (for example, $\{|0\rangle, |1\rangle\}$).

4. **Wait.** Necessary time of inaction on qubits in order to synchronize the operation of gates [1].

Errors can occur at any location spatially and temporally. If we're considering a single gate on a single qubit, the error might be easily containable using simple quantum error correction methods. But what makes the problem much more complicated is the error propagation. One error can be easily propagated both spatially (from one qubit to many qubits) and temporally (accumulative from one quantum operation to all consecutive operations). Therefore, we must make a design that error propagation can get controlled throughout the entire quantum circuit [1].

II. FAULT-TOLERANT QUANTUM CIRCUITS

A circuit with such design is called a fault-tolerant circuit. It does not completely remove or correct the errors but if given the error rate is not too high, it can contain the error propagation and makes us able to correct the resulting errors (with some existing quantum error correction techniques).

In Gottesman's paper [1], each of such construction is called a *gadget*, which simulate the behavior of the corresponding non-fault-tolerant action but instead acting on physical qubits, it acts on the logical qubits encoded in a quantum EC (error-correcting) code.

A. Basic Properties

There for two basic properties for a fault-tolerant gadget (when there are not too many errors in the input state and during the course):

1. Errors should be contained in the output state.

2. The gadget should perform the correct logical operation on the encoded state.

Detailed definitions are given in Gottesman's paper [1]. One construction that satisfies these properties is called transversal gates. The basic idea of transversal gates is to spread out any increase in errors among multiple blocks of the code, so that each block can have a controllable amount of errors. In particular, a gate that is constructed as a tensor product is a transversal gate. For example, $\text{CNOT}^{\otimes 7}$ acting on any CSS code implements the logical operation $\overline{\text{CNOT}}$ [7]. In fact, the entire logical Clifford group can be implemented transversally [1]. But the Clifford group is not sufficient for a universal quantum computation, and, on the other hand, no code would allows a universal set of transversal gates [8].

B. Error Correction and Measurements

1. *Shor Error Correction*

Shor error correction is a method to make error corrections in the quantum circuits fault-tolerant [7]. In short, we need to create many cat states via some non-fault-tolerant procedure and then verify pairs of qubits to see if they are the same. Each cat state can be used to measure one bit of the error syndrome and we would be able to measure the full error syndrome a couple of times. Then, from here, we can deduce a consensus error syndrome from the multiple times we measured and correct that error. This procedure can actually work for any stabilizer code [1]. However, it can be easily seen that Shor error correction is not sufficient as it requires to create many cat states (which means many extra gates) and a very low error rate since if it's high we can not find a consensus syndrome nor correct the final error.

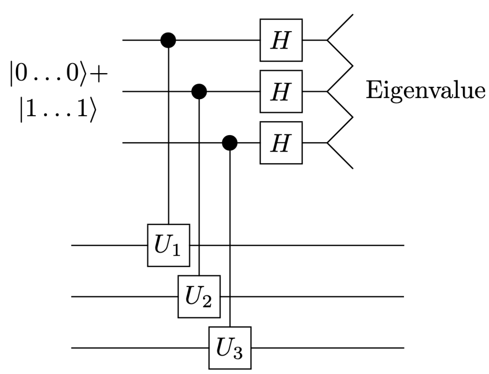

FIG. 1. A component of a fault-tolerant implementation of Shor error correction [1].

2. Steane Error Correction

Steane error correction is an improved method in efficiency but trades off the compatibility on non-CSS codes (that is, it only works on CSS codes) [1]. Recall the transversal CNOT appliess the logical $\overline{\text{CNOT}}$ on any CSS code. Then the procedure of Steane error correction goes as follows [1]:

- First we need to create an ancilla block $|\bar{0}\rangle + |\bar{1}\rangle$ in a codeword of the CSS code, and do a transversal CNOT from the data block to the ancilla block. In this way, it propagates the bit flip errors from the data block to the ancilla block without changing the encoded data. Then we can measure all the qubits in the ancilla block and treat the result as a classical codeword for C_1. In this way, we deduce the location of errors and correct them in the data block.

- A similar procedure follows for correcting phase errors: we create an ancilla block in the state $|\bar{0}\rangle$ and perform a transversal CNOT with the ancilla block as a control and the data block as a target. In this way, the phas errors are copied to the ancilla block, and we can measure them in the Hadamard-rotated basis, treating the measurement output as a classical codeword for C_2 with some errors. Then we can deduce the location of the phase errors using the classical decoding procedure for C_2 and correct them in the data block.

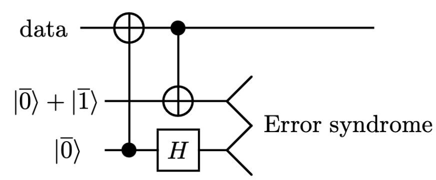

FIG. 2. A component that implements Steane Error Correction. [1].

3. Knill Error Correction

Knill error correction is another fault-tolerant error correction that works for any stabilizer code [9]. Similarly as Steane error correction, it also uses an ancilla state that is encoded using the same code as the data block. The basic idea is to use quantum teleportation to move the encoded state into different blocks of the code. The bell measurement would gain more information than needed for teleporation, and the extra information can be used to tell the error syndrome of the combined errors [9]. It can also be easily modified to a fault-tolerant measurement gadget by substituting an ancilla in the state $|\bar{0}\rangle$ for the encoded EPR pair in the error correction circuit [1].

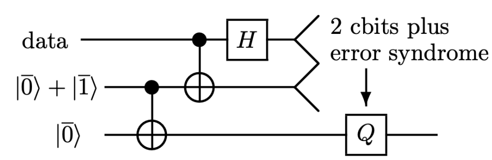

FIG. 3. A component that implements Knill Error Correction. [1].

Both Steane and Knill error corrections rely on moving the error correction step into the creation of a particular ancilla state. Advantages of doing so include not working directly on the data can reduce the time of waiting and eventually reduce the amount of errors accumulated during the wait time, and the ability to verify the ancilla states since they are created in known states [10].

C. Gates

As mentioned in Section II A, the Clifford group, which can be made fault-tolerant via transversal gates, does not form a universal set of fault-tolerant gates [1]. We define a set

$$C_k = \{U \mid UQU^\dagger \in C_{k-1} \text{ for all } Q \in C_1\} \quad (1)$$

where C_1 is the Pauli group $\mathcal{P}_n$ and C_2 is the Clifford group $\mathcal{C}_1$. Then using the idea in Figure 4, with a teleporation construction and appropriate ancilla states, we would be able to perform a gate from C_k once we know how to perform gate from C_{k-1} [1].

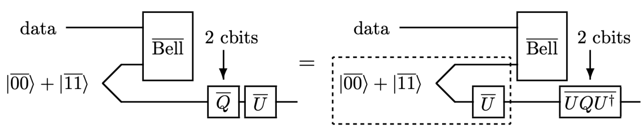

FIG. 4. A teleportation construction that shows how to perform $\overline{UQU^\dagger}$ given we know hot o perform $\overline{U}$ [1].

In fact, for the 7-qubit code and some similar CSS codes, the Clifford group plus $R_{\pi/8} \in C_3$ already form a universal set of fault-tolerant gates. Any other gate can actually be approximated with gates from C_k as arbitraily close as we wish following a consequence of the Solovay-Kitaev theorem [11].

D. State Preparation

State preparation is the last one of the four components of quantum circuit we mentioned in Section I A that need to make fault-tolerant. Most of the methods mentioned in Section II B rely on the ability to fault-tolerantly prepare some ancilla states [1]. For example, to perform Steane or Knill error correction, we need to prepare the ancilla states $|\bar{0}\rangle$ and/or $|\bar{0}\rangle + |\bar{1}\rangle$ without many errors. One way to do this is through a version of Shor error correction. In short, we use the Steane error correction procedure to detect errors instead of correcting them. if errors are detected, we discard the main state and measured the ancilla state. By repeating this procedure using the ancilla states that passed previous rounds of screening, with a sufficient number of iterations, this is effectively a fault-tolerant state preparation procedure, even for large distance codes [1].

E. Error Rate

All of the fault tolerance designs above serve for a common goal: to reduce the error rate for quantum circuits. In fact, it can be shown that for any local stochastic model less than a threshold error rate p_T, there exists a compatible fault-tolerant circuit [1]. For example, for the 7-qubit code, one find $p_T \geq 2.73 \times 10^{-5}$ [12] [13]. With elevated ancilla preparation techniques, Knill was able to, through simulation, achieve a threshold p_T of as high as 5% [9]. The threshold can be further improved in various ways including introducing systematic error (for example, a R_θ phase rotation over a consistent small angle), devising gadgets which only involve nearest-neighbor interactions, and utilizing asymmetry between X and Z errors [1].

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