Measurement-based Computation with Symmetry Protected Phases in 1-D

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This paper attempt to explore the computational potential of symmetry protected topological phases in 1-D. Measurement based computation is used with tensor product representation of resource states. The work from Stephen et al. established a direct correlation between mathematical modeling of SPT phases and the quantum computation power of resource state within them(in the 1-D case).

INTRODUCTION

Measurement-bases quantum computation has been well studied for dimensional cases larger than 1. It had been shown that to simulate a quantum circuit in D dimension with MBQC, we need an entangled resource state in $D+1$ dimension, hence the 1-D resource states had been overlooked while using MBQC scheme.

Symmetry-protected topological(SPT) in 1-D

For (finite)gapped systems, two ground states are said to be of same phase protected by symmetry $\mathbb{S}$ if they can be deformed into each other without breaking the symmetry and closing the ground state energy gap. An example of SPT order that we will be focusing on is the 1-D AKLT state shown in lecture. The state is protected by $Z_2 \times Z_2$ symmetry(π-rotation about x,y,z axes).

Measurement-based quantum computation(MBQC)

MBQC is a method that simulates a quantum circuit by performing local measurement on single qbits on a entangled state. Computation with such method consumes entanglement(by measurements) to allow simulation of unitary evolution. The backbone of the computational method is the resource state prepared for measurement operations. One classic example of such state is the 2-D cluster state to which we can mold many unitary gates onto the "substrate" with measurement patterns[1]. The 2-D cluster state has the ability to simulate every 1-D quantum circuits, meaning it's a resource state for universal MBQC in 1-D.

MBQC ON 1-D AKLT STATE

Setup

For description of 1-D AKLT state from spin chains, we will sue the Matrix Product State(MPS) formalism introduced on page 120-121 of the lecture notes. A state $|\psi\rangle$ of a finite sized spin chain can be represented as following

$$|\psi\rangle = \sum_{i_1,\ldots,i_N} \langle R| A^{i_N} A^{i_{N-1}} \ldots A^{i_1} |L\rangle |i_1 \ldots i_N\rangle \quad (0.1)$$

Here, $|L\rangle$ and $|R\rangle$ vectors encode the boundary conditions of the chain, index N labels each site with local dimension d(same for each site), and A is the tensor at site N with rank $3(2\times2\times3$, projection from spin 1/2 pair to spin 1).

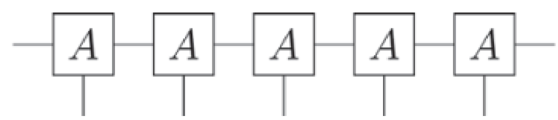

FIG. 1: graph representation of the tensor product state with rank-3 tensor A on each site

From Measurement to Unitary Operation

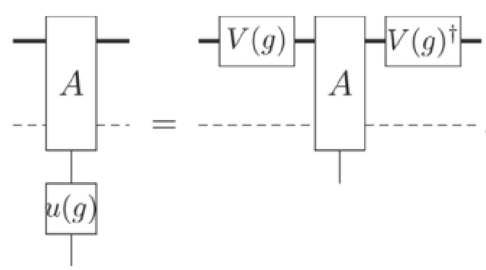

FIG. 2: Note that the subsystem connected to tensor A(dotted lines) but not protected by the on-site $Z_2 \times Z_2$ symmetry is considered "junk" subspace

When we perform a measurement on one site with a set of orthogonal basis, the projection operation reduces the on-site tensor to rank-2, which could be seen as operators in correlation space. The exact operator imposed onto the link states depend on the measurement out come, hence it's not deterministic but rather probabilistic.

One important mechanism is the propagation of such product of measurement along the chain in correlations space. A tensor appearing the the MPS representation of ground states in the symmetry-protected phase satisfies the condition shown in figure 2. Using this transformation we can show that the operator resulted from measurement on previous site can be propagated through the next site with transformation $u(g)$ at 'base.' On a different perspective, such change can be absorbed my measurement on subsequent sites with transformed basis(that within the symmetry group).

The point is that the operations that is propagating in the correlation space, connecting the two edge states, is determined by the the set of basis and results of each site measurement in the bulk. Practically, measurement outcome at each site determines the basis of measurement on the following site. As such, the operator generated from the left-most site measurement could propagate to the right-most site 'unharmed' and act on the edge state.

You can find a more rigorous description and proofs of the above manipulations to the tensor product representations in Else's work[2].

Example

For the AKLT model, the projector on each spin-1 site is

$$|0\rangle \frac{\sigma_z}{\sqrt{2}} + |-1\rangle \sigma^+ + |1\rangle (-\sigma^-)$$

the operators resulting from measurement will simply be the three matrices $\{\frac{\sigma_z}{\sqrt{2}}, \sigma^+, -\sigma^-\}$ listed above, if we measure in the basis of $\{|x\rangle, |y\rangle, |z\rangle\} = \{|+1\rangle, |-1\rangle, |0\rangle\}$.

If we had changed the measurement basis to $\{(|-1\rangle - |+1\rangle)/\sqrt{2}, i(|-1\rangle + |+1\rangle)/\sqrt{2}, |0\rangle\}$, the resulting operators in the correlation space is simply the set of XYZ Pauli operators. If we measure each site from the left, the accumulated operation on the right edge state will be the sequence of Pauli operators according to the sequence of measurement outcomes. Such basis of measurement is called the wire basis.

It's now easier to develop a measurement basis that produce more general unitary operations on the correlation space. A rotation in our measurement basis(wire basis) about the $|z\rangle$ axis produced the following basis

$$\{\cos\phi\,|x\rangle + \sin\phi\,|y\rangle, \cos\phi\,|y\rangle - \sin\phi\,|x\rangle, |z\rangle\}$$

per symmetry shown in figure 2, the set of resulting set of operators(somewhat intuitively) are

$$\{\sigma_x e^{i\phi\sigma_z}, \sigma_y e^{i\phi\sigma_z}, \sigma_z \mathbb{1}\}$$

The operation could be regard as set of unitary rotations $\{e^{i\phi\sigma_z}, e^{i\phi\sigma_z}, \mathbb{1}\}$ and a set of by-products $\{\sigma_x, \sigma_y, \sigma_z\}$. If we only want to impose unitary rotations on the edge state, we'll need to propagate the byproducts produced at each site by transforming the basis of measurement at subsequent sites. This also implies temporal order in the measurements.

However, the randomness of the measurement outcomes means if we carry out the measurements described in the example above, the resulting state after byproduct propagation and absorption will behave like a 'mixed state' rather than unitary rotation from the original. However, if we choose to rotate only by and infinitesimal angle $d\phi$, the resulting 'mixed' state is then approximated by[3]

$$\boldsymbol{\rho} = e^{-ipd\phi\sigma_z} |\Psi\rangle \langle\Psi| e^{ipd\phi\sigma_z} + \mathcal{O}(d\theta^2) \qquad (0.2)$$

Where the modifying term p is determined by the success rate of our unitary rotation(with equal probable outcome in our example $p = 2/3$). This means for finite angle rotation to work, we have to cascade those infinitesimal rotations one after another while propagating byproducts from all of the sites. With some tweak to the measurement basis in the example, we can also rotate about the x axis. This effectively give us construction for all operations with the group $SU(2)$. The cost of performing logical operations in the correlation space is the amount of site that needs to be measured(the amount of entanglement being consumed) that is inversely proportional to the success rate p.

SYMMETRY IS THE ENABLER

The mechanism of generating unitary operators relies on the on-site symmetry $G(Z_2 \times Z_2$ in the case of AKLT) and cohomology class $[\omega] \in H^2(G, U(1))$ that describes how the on-site symmetry transform to the correlation space. Here's the theorem proposed by Stephen et al.[3]

Consider an SPT phase defined by an on-site symmetry group G and cohomology class $[\omega]$. Suppose there exists a finite abelian subgroup $H \subset G$ such that $[\omega|_H]$ is maximally non-commutative, and let p^n be a prime power dividing $\sqrt{|H|}$. Then the lie group that uniformly defines the computational power of each resource state has $\mathcal{L}[\mathcal{O}] \supset SU(p^n)$.

The theorem, which is proven in [3], implies a lower bound on the compuational power of resources state that has phase protected by G and shows the usefulness of such 1-D SPT phases in the scheme of MBQC.

ACKNOWLEDGEMENTS

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