University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2025
Assignment 2 – Solutions

Due 11:59pm Tuesday, January 21, 2025

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1. **The vacuum is a fluid with $p = -\rho$.**

   We said in lecture that the vacuum energy density ρ gravitates and that, when positive, its effect is to cause space to inflate – to expand exponentially in time. An important aspect of this phenomenon is that the vacuum fluctuations produce not only an energy density, but a *pressure*, $p = T^i_i$ (no sum on i), of the form $p = -\rho$, which is negative for $\rho > 0$. The vacuum therefore acts as a perfect fluid with $p = -\rho$. (The stress tensor for a perfect fluid in terms of its velocity field u^μ takes the form $T^{\mu\nu} = (p+\rho)u^\mu u^\nu + pg^{\mu\nu}$, so in a frame with $u^\mu = (1, \vec{0}^\mu)$, $T^0_0 = \rho, T^i_i = p$.) Solving Einstein's equations with such a source produces an inflating universe. In this problem we show that this is the case from QFT.

   (a) Show that the energy-momentum tensor for a free relativistic scalar field ($S[\phi] = \int d^D x \sqrt{g}\mathcal{L}, \mathcal{L} = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{m^2}{2}\phi^2$) takes the form

   $$T_{\mu\nu} = a\partial_\mu\phi\partial_\nu\phi - bg_{\mu\nu}\mathcal{L}$$

   with some constants a, b.

   You may do this either by deriving the Noether currents for spacetime translations, or by extracting the response to a variation of the spacetime metric, $T_{\mu\nu}(x) = \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}(x)}$. Here and above $\sqrt{g} \equiv \sqrt{|\det g|}$.

   (b) Reproduce the formal expression for the vacuum energy

   $$\langle 0|\mathbf{H}|0\rangle = V \int đ^d k \frac{1}{2}\hbar\omega_{\vec{k}}$$

   using the two point function

   $$\langle 0|\, \phi(x)^2\, |0\rangle = \langle 0|\phi(0)\phi(0)|0\rangle = \lim_{\vec{x},t\to 0} \langle 0|\, \phi(x)\phi(0)\, |0\rangle$$

   and its derivatives. (V is the volume of space.)

   Some details of this calculation appear in Zee's book, section I.8, around equation (19). Or see below.

(c) Show that the vacuum expectation value of the pressure

$$\langle 0|T_{ii}|0\rangle$$

(no sum on i) gives the same answer, up to a sign.

[Hints: You'll find a quite different looking integral from the vacuum energy. Use rotation invariance of the vacuum to simplify the answer. The claim is that however you regulate the integral for the vacuum pressure and $\frac{1}{2}\int đ^d k\omega_k$, you'll get the same answer (as long as the regulator respects the symmetries). A convenient regulator is *dimensional regularization*: simply treat the dimension d as an arbitrary complex number.]

(d) Argue that $p = -\rho$ is required in order that the vacuum energy does not specify a preferred rest frame.

(e) Evaluate the vacuum energy using the Feynman rules. That is, draw this amplitude as a Feynman diagram which is a circle – a line connecting a point to itself – with an operator insertion at the point.

(f) [bonus problem] Show that the resulting vacuum energy momentum tensor ($T_{00} = \rho, T_{ii} = -\rho$ (no sum on i)) is the same as the contribution to the energy-momentum tensor from an action of the form

$$S_{\rm cc} = \int d^D x \sqrt{g}\Lambda$$

where Λ is a constant (the cosmological constant).

If you wish, plug in the FRW ansatz for the metric $ds^2 = -dt^2 + a(t)^2 d\vec{x}^2$ and show that Einstein's equations in the presence of a positive cosmological constant

$$\frac{\delta S[g]}{\delta g_{\mu\nu}(x)} = 0, \text{ with } S[g] = \frac{1}{16\pi G_N}\int d^D x\sqrt{g}R + S_{\rm cc} \tag{1}$$

have the solution $a(t) = e^{Ht}$ for some H determined by Λ and G_N.

I didn't answer the questions quite in order, but they are all here:

$$T^\mu_\nu = \partial^\mu\phi\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}.$$

$$T_{00} = \dot{\phi}^2 - \mathcal{L} = \mathfrak{h} = \frac{1}{2}\left(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right).$$

$$T_{ii} = (\partial_i\phi)^2 + \frac{1}{2}\left(-\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2)\right) \quad \text{(no sum on } i).$$

We use the fact that the vacuum is translation invariant to identify

$$\int d^dx \langle 0| T^\mu_\nu(x) |0\rangle = L^d \langle 0| T^\mu_\nu(0) |0\rangle$$

where L^d is the volume of space. The following consequences of the mode expansion of the scalar field will be useful:

$$\int d^dx m^2\phi^2 = \int_x \int_k \int_q \frac{m^2}{\sqrt{4\omega_k\omega_q}} \left(a_k e^{i\vec{k}\cdot\vec{x}} + a_k^\dagger e^{-i\vec{k}\cdot\vec{x}}\right) \left(a_q e^{i\vec{q}\cdot\vec{x}} + a_q^\dagger e^{-i\vec{q}\cdot\vec{x}}\right) \tag{2}$$

$$= \int đ^dk \frac{m^2}{2\omega_k} \left(a_k a_k^\dagger + a_k^\dagger a_k + (a_k a_{-k} + h.c.)\right). \tag{3}$$

$$\int d^dx \pi^2 = (-\mathbf{i})^2 \int_x \int_k \int_q \sqrt{\frac{\omega_k\omega_q}{4}} \left(a_k e^{i\vec{k}\cdot\vec{x}} - a_k^\dagger e^{-i\vec{k}\cdot\vec{x}}\right) \left(a_q e^{i\vec{q}\cdot\vec{x}} - a_q^\dagger e^{-i\vec{q}\cdot\vec{x}}\right) \tag{4}$$

$$= \int đ^dk \frac{\omega_k}{2} \left(a_k a_k^\dagger + a_k^\dagger a_k - (a_k a_{-k} + h.c.)\right). \tag{5}$$

$$\int d^dx (\vec{\nabla}\phi)^2 = \int_x \int_k \int_q \sqrt{\frac{1}{4\omega_k\omega_q}} \left(\mathbf{i}k a_k e^{i\vec{k}\cdot\vec{x}} - \mathbf{i}k a_k^\dagger e^{-i\vec{k}\cdot\vec{x}}\right) \left(\mathbf{i}q a_q e^{i\vec{q}\cdot\vec{x}} - \mathbf{i}q a_q^\dagger e^{-i\vec{q}\cdot\vec{x}}\right) \tag{6}$$

$$= \int đ^dk \frac{k^2}{2\omega_k} \left(a_k a_k^\dagger + a_k^\dagger a_k + (a_k a_{-k} + h.c.)\right). \tag{7}$$

When taking vacuum expectation values of these objects, only the first term survives:

$$\langle 0| (a_k a_k^\dagger + a_k^\dagger a_k) |0\rangle = \langle 0| (2a_k^\dagger a_k + 1) |0\rangle = 1.$$

Therefore the energy density is

$$\langle 0| T_{00} |0\rangle = \frac{1}{2} \int đ^dk \omega_k$$

and the pressure is

$$\langle 0| T_{ii} |0\rangle = \frac{1}{2} \int đ^dk \frac{k_i^2}{\omega_k} \qquad \text{(no sum on } i\text{)}.$$

The expression we find for the vacuum pressure looks quite different from the vacuum energy. But there are a few observations about symmetry that help to bring out their commonalities:

(a) By Lorentz invariance of the vacuum, the vacuum expectation value of the stress tensor can only have the form

$$\langle 0| T_{\mu\nu} |0\rangle = \Lambda \eta_{\mu\nu}$$

because the Minkowski metric is the only Lorentz invariant 2-index object available. This has the form of the stress tensor of a perfect with $p = -\rho$. (Some interesting properties of a fluid of this form are described here.) Since it's Lorentz invariant, with this form of the pressure, the vacuum energy does not specify a preferred frame.

(b) The vacuum is also isotropic, meaning that $\langle 0| T_{ii} |0\rangle$ (no sum) must be independent of i. We can use this to write the pressure in a more symmetric-looking form. Since it's equal to its average over $i = 1..d$, we can replace $k_i^2 \rightsquigarrow \vec{k}^2/d$ in the integrand:

$$\langle 0| T_{ii} |0\rangle = \frac{1}{2}\frac{1}{d}\int d^d k \frac{\vec{k}^2}{\omega_k} \qquad \text{(no sum on } i\text{).}$$

Thus we find

$$\langle 0| T_{00} |0\rangle = \frac{1}{2}\int d^d k \omega_k \equiv \rho \tag{8}$$

$$\langle 0| T_{ii} |0\rangle = \frac{1}{2}\frac{1}{d}\int d^d k \frac{\vec{k}^2}{\omega_k} \equiv p \qquad \text{(no sum on } i\text{).} \tag{9}$$

The resulting (divergent) integrals aren't obviously the same. The claim, based on Lorentz symmetry, is that if they are regularized in a Lorentz-invariant way they will always give the same answer.

Here is an example of a very symmetrical way to regularize them, where we can see their equality explicitly: simply interpret d as an arbitrary complex number. (This is called *dimensional regularization*.)

$$\rho = \int đ^d k \sqrt{\vec{k}^2 + \omega^2} = \underbrace{\frac{\Omega_d}{(2\pi)^d}}_{\equiv K_d} \int_0^\infty dk k^{d-1} \sqrt{k^2 + m^2} = -\frac{1}{2} K_d m^{d+1} \int_0^1 dx x^a (1-x)^b$$

with $a = -\frac{d+3}{2}, b = \frac{d-2}{2}$. In the last equality we made the change of variables $x \equiv \frac{m^2}{k^2+m^2}$ (so $k = \sqrt{\frac{1-x}{x}}, dk = -\frac{1}{2}\frac{dx}{x^{3/2}(1-x)^{1/2}}, \sqrt{k^2+m^2} = \frac{m}{\sqrt{x}}$). Now using the identity

$$\int_0^1 dx x^a (1-x)^b = \frac{\Gamma(1+a)\Gamma(1+b)}{\Gamma(2+a+b)}$$

where $\Gamma(x)$ is the Euler Gamma function (and this combination is related to the Euler Beta function), we have

$$\rho = -\frac{1}{2} K_d m^{d+1} \frac{\Gamma(1+a)\Gamma(1+b)}{\Gamma(2+a+b)}.$$

On the other hand, using rotation invariance,

$$p = \frac{1}{d}\int đ^d k \frac{k^2}{\sqrt{\vec{k}^2+\omega^2}} = \frac{1}{d}\underbrace{\frac{\Omega_d}{(2\pi)^d}}_{\equiv K_d}\int_0^\infty dk \frac{k^{d+1}}{\sqrt{k^2+m^2}} = -\frac{1}{2}K_d m^{d+1}\int_0^1 dx x^a (1-x)^{b+1}.$$

But the Gamma function satisfies the (factorial) identity $\Gamma(1+x) = x\Gamma(x)$, and therefore we have

$$p = \rho \frac{1}{d}\frac{1+b}{2+a+b} = -\rho.$$

It may be tempting to instead use a hard cutoff on the spatial momentum, but this isn't Lorentz invariant! (In a different rest frame, such a cutoff will look different.) So it breaks the relation we are trying to show.

2. **Casimir force is regulator-independent.** [Bonus problem] Suppose we use a different regulator for the sum in the vacuum energy $\sum_j \hbar\omega_j$. The regulator we'll use here is an analog of Pauli-Villars. In the notation introduced in the lecture notes, we replace

$$f(d) \rightsquigarrow \frac{1}{2}\sum_{j=1}^{\infty}\omega_j K(\omega_j)$$

where the function K is

$$K(\omega) = \sum_\alpha c_\alpha \frac{\Lambda_\alpha}{\omega+\Lambda_\alpha}.$$

We impose two conditions on the parameters c_α, Λ_α:

- We want the low-frequency answer to be unmodified:

$$K(\omega) \overset{\omega\to 0}{\to} 1$$

– this requires $\sum_\alpha c_\alpha = 1$.

- We want the sum over j to converge; this requires that $K(\omega)$ falls off faster than ω^{-2}. Taylor expanding in the limit $\omega \gg \Lambda_\alpha$, we have

$$K(\omega) \overset{\omega\to\infty}{\to} \frac{1}{\omega}\sum_\alpha c_\alpha\Lambda_\alpha - \frac{1}{\omega^2}\sum_\alpha c_\alpha \Lambda_\alpha^2 + \cdots .$$

So we also require $\sum_\alpha c_\alpha\Lambda_\alpha = 0$ and $\sum_\alpha c_\alpha \Lambda_\alpha^2 = 0$.

First, verify the previous claims about $K(\omega)$.

Then compute $f(d)$ and show that with these assumptions, the Casimir force is independent of the parameters c_α, Λ_α.

[A hint for doing the sum: use the identity

$$\frac{1}{X} = \int_0^\infty ds e^{-sX}$$

inside the sum to make it a geometric series. To do the remaining integral over s, Taylor expand the integrand in the regime of interest.]

This problem comes from Zee, *QFT in a nutshell*, 2d edition pages 74-75.

3. **Casimir energy from balls and springs.** [More difficult bonus problem] Regularize the Casimir energy of a 1d scalar field by discretizing space. If you suppose there are $N \equiv d/a \in \mathbb{Z}$ lattice points in the left cavity

$$| \leftarrow d \rightarrow | \longleftarrow \qquad L - d \qquad \longrightarrow |$$

what answer do you find for the force on the middle plate?

[Hint: you will find the wrong answer! The problem is that with these assumptions d cannot vary continuously. One way to allow d to vary continuously (and get the right answer) is to impose $\phi(0) = 0 = \phi(d)$, but do not assume d corresponds to a lattice site.]

Using the set-up from lecture, the vacuum energy is $E_0(d) = f(d) + f(L-d)$ with

$$f(d) = \frac{1}{2}\hbar \sum_{j=1}^{N} \omega_j \tag{10}$$

with

$$\omega_j = 2\sqrt{\frac{\kappa}{m}} \sin\frac{j\pi}{N} = \frac{2c}{a}\sin\frac{j\pi}{N}, \quad N \equiv d/a \in \mathbb{Z}, j = 1..N \tag{11}$$

where we used the relation between the wave speed and the spring constant. Note that we are interested in $N \gg 1$ to approach something like a continuum. The sum is (it is a sum of two geometric series, or ask Mathematica)

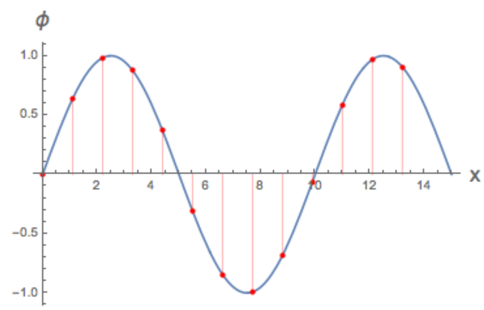
$$\sum_{j=1}^{N} \sin\frac{j\pi}{N} = \cot\frac{\pi}{2N} \overset{N\gg 1}{=} \frac{2N}{\pi} - \frac{\pi}{6N} + \mathcal{O}(N^{-2}).$$

Then

$$E_0 = f(d) + f(L-d) \overset{a \ll d}{=} \frac{1}{2}\frac{2c}{a}\left(\frac{2}{\pi}\frac{d + L - d}{a} - \frac{\pi}{6}\left(\frac{a}{d} + \frac{a}{L-d}\right) + \mathcal{O}(a^2)\right)$$

As before the terms that have negative powers of a are independent of d and therefore do not contribute to the force. The terms with more powers of a go to zero in the limit $d/a \to \infty$. The only term that survives is the term independent of the cutoff a, which produces an attractive force that goes like $\frac{\pi}{d^2}$. The only problem is that the coefficient is 1/6 rather than 1/24!

This innocent-looking factor of four hides a deeper problem with the calculation I just described: N must be an integer, but I've replaced it with d/a which we are trying to vary continuously. It is actually possible to let d/a vary continuously, at the cost of putting boundary conditions $\phi(x=0)=0$ and $\phi(x=d)=0$ at locations that are not on the grid. That is, let the lattice sites be $x_j = ja$, but don't assume that d is an integer multiple of a. So $\phi(0)=0$ says $\phi(x) \propto \sin kx$ for some k. $\phi(d)=0$ says $kd \in \pi\mathbb{Z}$ so $k = \pi j/d, j = 1..N$. But d need not be Na! Like this (here $r = 15, a = 1.1$, and I've drawn the $j = 3$ mode):

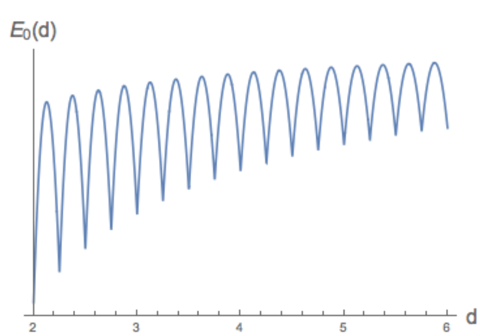


(Notice that this is harder to do for periodic boundary conditions.)

Then $\omega_k = \frac{2c}{a} \sin \frac{ka}{2}$ as before, with $k = \frac{\pi j}{d}, j = 1..N$. Then

$$f(r) = \frac{1}{2}\sum_j \omega_j = \frac{c}{a}\sum_{j=1}^{N} \sin\frac{\pi j a}{r} = \frac{c}{2a}\csc\left(\frac{\pi}{2z}\right)\left(\sin\left(\frac{(z-1)\pi}{2z}\right) - \sin\left(\frac{(z-1-2N)\pi}{2z}\right)\right),$$

where N is the number of lattice sites and $z \equiv r/a$ is the length of the interval in units of the lattice spacing, *which need not be an integer.* So $N = \lfloor r/a \rfloor$, the *floor* of (largest integer less than) r/a. The total energy in left and right cavities $E_0(d) = f(d) + f(L-d)$ then looks like this:



where I've set $c = 1, L = 1000, a = \frac{1}{4}$, just as in Matt Schwartz's plot in §15.2. Using the following expansions for large intervals (large $z = r/a$):

$$\csc\frac{\pi}{2z} = \frac{2z}{\pi} + \frac{\pi}{12z} + \mathcal{O}(z^{-3})$$

$$\sin\left(\frac{(z-1)\pi}{2z}\right) = 1 - \frac{\pi^2}{8z^2} + \mathcal{O}(z^{-4})$$

and, using $\frac{\lfloor z \rfloor}{z} = 1 - \frac{x}{z}$ where $x \equiv z - \lfloor z \rfloor$,

$$\sin\left(\frac{(z-1-2N)\pi}{2z}\right) = \sin\left(\left(\frac{-\pi}{2}\right)(1 + \frac{1-2x}{z})\right) \tag{12}$$

$$= -1 + \frac{\pi^2}{8z^2}(1-2x)^2 + \mathcal{O}(z^{-4}) \tag{13}$$

$$\overset{\text{average}}{\rightsquigarrow} -1 + \frac{\pi^2}{8z^2}\frac{1}{3} + \mathcal{O}(z^{-4}) \ . \tag{14}$$

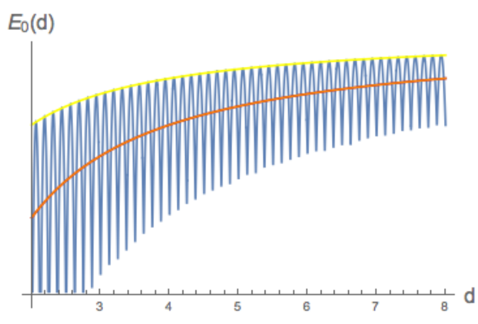
In the last step we averaged over the fluctuations of x, which become more and more rapid as the lattice spacing shrinks. Putting these together, we get

$$f(r) = \frac{c}{2a}\left(\frac{2z}{\pi} + \frac{\pi}{12z} + \cdots\right)\left(1 - \frac{\pi^2}{8z^2} + \cdots - \left(-1 + \frac{\pi^2}{8z^2}\frac{1}{3} + \cdots\right)\right) \tag{15}$$

$$= \frac{r}{\pi a^2} - \frac{\pi}{24r} + \mathcal{O}(a/r)^2 \tag{16}$$

which is exactly the answer we got from other regulators.

Comparing to the wildly-oscillating plot above, the averaged result in (16) is the orange curve here:



The yellow curve is what we get if we just set the $(1-2x)^2$ in (13) to zero.