University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2025
Assignment 3

Due 11:59pm Tuesday, January 28, 2025

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1. **Brain-warmer.**

   Use the Clifford algebra to show that in 3+1 dimensions

   $$\gamma^\mu \left(x\not{p} + m\right)\gamma_\mu = -2x\not{p} + 4m$$

   where as usual $\not{p} \equiv p^\mu \gamma_\mu$. This identity will be useful in the numerator of the electron self-energy.

2. **An example of renormalization in classical physics.**

   Consider a classical scalar field in $D+2$ spacetime dimensions coupled to an *impurity* (or defect or brane) in D dimensions, located at $X = (x^\mu, 0, 0)$. Suppose the field has a self-interaction which is localized on the defect. For definiteness and calculability, we'll consider the simple (quadratic) action

   $$S[\phi] = \int d^{D+2}X \left(\frac{1}{2}\partial_M \phi(X)\partial^M \phi(X) + g\delta^2(\vec{x}_\perp)\phi^2(X)\right).$$

   Here $X^M = (x^\mu, x_\perp^i)$, $\mu = 0..D-1$, $i = 1, 2$, *i.e.* $x_\perp$ are coordinates transverse to the impurity.

   (a) What is the mass dimension of the coupling g? This is why I picked a codimension[1]-two defect.

   (b) Find the equation of motion for ϕ. Where have you seen an equation like this before?

   (c) We will study the propagator for the field in a mixed representation:

   $$G_k(x,y) \equiv \langle \phi(k,x)\phi(-k,y)\rangle = \int d^D z\; e^{\mathbf{i}k_\mu z^\mu} \langle \phi(z,x)\phi(0,y)\rangle$$

   – *i.e.* we go to momentum space in the directions in which translation symmetry is preserved by the defect. Find and evaluate the diagrams contributing to $G_k(x,y)$ in terms of the free propagator $D_k(x,y) \equiv \langle \phi(k,x)\phi(-k,y)\rangle_{g=0}$. (We will not need the full form of $D_k(x,y)$.) Note that there are no loop diagrams, and in this sense, all the physics here is classical. Sum the series. I found it convenient to do this problem in Euclidean spacetime, so G and D are Euclidean propagators.

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[1] An object whose position requires specification of p coordinates has codimension p.

(d) You should find that your answer to part 2c depends on $D_k(0,0)$, which is divergent. This divergence arises from the fact that we are treating the defect as infinitely thin, as a pointlike object – the δ^2-function in the interaction involves arbitrarily short wavelengths. In general, as usual, we must really be agnostic about the short-distance structure of things. To reflect this, we introduce a regulator. For example, we can replace the fourier representation of $D_k(0,0)$ with the cutoff version

$$D_k(0,0;\Lambda) = \int_0^\Lambda đ^2 q \frac{e^{\mathbf{i}q\cdot 0}}{k^2+q^2}. \tag{1}$$

Do the integral.

(e) Now we renormalize. We will let the *bare coupling* g (the one which appears in the Lagrangian, and in the series from part 2c) depend on the cutoff $g = g(\Lambda)$. We wish to eliminate $g(\Lambda)$ in our expressions in favor of some measurable quantity. To do this, we impose a renormalization condition: choose some reference scale μ, and demand that[2]

$$G_\mu(x,y) \stackrel{!}{=} D_\mu(x,y) - g(\mu) D_\mu(x,0) D_\mu(0,y). \tag{2}$$

This equation defines $g(\mu)$, which we regard as a physical quantity. Show that (2) is satisfied if we let the bare coupling be $g(\Lambda) = g(\mu)Z$, with

$$Z = \frac{1}{1 - \frac{g(\mu)}{4\pi}\ln\left(\frac{\Lambda^2}{\mu^2}\right)}.$$

(f) Find the beta function for g,

$$\beta_g(g) \equiv \mu \frac{dg(\mu)}{d\mu},$$

and solve the resulting RG equation for $g(\mu)$ in terms of some initial condition $g(\mu_0)$. Does the coupling get weaker or stronger in the UV?

3. **Scale invariance in QFT in $D = 0+0$, part 1.** [I got this problem from Frederik Denef.]

A nice realization of QFT in $0+0$ dimensions is the statistical mechanics of a collection of non-interacting particles. The canonical partition function for a single particle (moving in one dimension) is

$$Z = \int đP dX e^{-\beta H} \propto \sqrt{T} Z_V(T) \tag{3}$$

[2] Note that if we worked in real time, there would be an extra $\mathbf{i}$ in front of the second term on the RHS.

with $H = \frac{P^2}{2} + V(X)$ and $T = 1/\beta$. The momentum integral is Gaussian and we can just do it. The partition function of N non-interacting indistinguishable particles is then $Z^N/N!$, which just multiplies the energy $U = T^2 \partial_T \log Z$ by a factor of N, so we don't miss anything by focussing on the single particle.

Let's consider the case

$$V(X) = aX^2 + bX^4 + cX^6 \tag{4}$$

and figure out the important features of the temperature dependence of the thermodynamic quantities by scaling arguments.

(a) Assuming $a \neq 0, b \neq 0, c \neq 0$, find the behavior of the thermal energy U and the heat capacity $C = \partial_T U$ in the limit $T \to 0$ and in the limit $T \to \infty$ using scaling arguments. Which parts of the potential determine the respective limiting behaviors?

(b) If some of the couplings a, b, c vanish, the low or high temperature scaling behavior may change. For example, what is the heat capacity at low temperature when $a = 0, b \neq 0$?

(c) When b is sufficiently large (and $a \neq 0, c \neq 0$), there will be an intermediate temperature regime over which the heat capacity is again constant, but different from the low- and high-temperature limits. What is this heat capacity?

(d) In general, we can think of the change of C with T as a kind of classical renormalization group (RG) flow, interpolating between 'fixed points' where C becomes constant. In general, these fixed points correspond to potentials $V(X)$ with a scaling symmetry $V(\lambda^\Delta X) = \lambda V(X)$ for some choice of scaling dimension Δ of X. What is the heat capacity for a fixed point with scaling dimension Δ for X?

(e) Borrowing more language of the renormalization group, we can classify deformations $\delta V(X) = \epsilon X^m$ of a fixed point $V(X) \propto X^{2n}$ as irrelevant, marginal, or relevant, depending on whether the deformation becomes dominant or negligible in the IR limit, *i.e.* in the limit of low T. Here and below ϵ can take on any value, not necessarily small. Restricting to deformations with an $X \to -X$ symmetry, what are the relevant and irrelevant deformations of $V(X) = X^{2n}$? (Note that a deformation $\delta V = \epsilon X^{2n}$ can be absorbed into a redefinition of X, which does not change the heat capacity.)

(f) The T-dependence of correlation functions (here, expectation values of powers of X) at fixed points is also determined by the scaling properties. What is the T-dependence of $\langle X^k \rangle$ at a fixed point $V(X) = X^{2n}$?

(g) Non-polynomial $V(X)$ can be considered as well. For example, what is the heat capacity at small and large T for $V(X) = (1 + X^2)^{1/n}$?