University of California at San Diego – Department of Physics – Prof. John McGreevy

# Physics 215B QFT Winter 2025
# Assignment 5 – Solutions

**Due 11:59pm Tuesday, February 11, 2025**

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1. **Brain-warmer.**

   Prove the Gordon identities

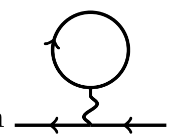
   $$\bar{u}_2 \left(q^\nu \sigma_{\mu\nu}\right) u_1 = \mathbf{i}\bar{u}_2 \left((p_1+p_2)_\mu - (m_1+m_2)\gamma_\mu\right) u_1$$

   and

   $$\bar{u}_2 \left((p_1+p_2)^\nu \sigma_{\mu\nu}\right) u_1 = \mathbf{i}\bar{u}_2 \left((p_2-p_1)_\mu - (m_2-m_1)\gamma_\mu\right) u_1$$

   where $q \equiv p_2 - p_1$ and $\not{p}_1 u_1 = m_1 u_1, \bar{u}_2 \not{p}_2 = m_2 \bar{u}_2$, using the definitions and the Clifford algebra.

2. **Tadpole diagrams.**

   (a) Why don't we worry about the following diagram  as a correction to the electron self-energy in QED?

   It has to vanish by Lorentz symmetry: the object  would be a source $j^\mu$ for the electromagnetic field in the vacuum. At one loop, we can check that $\int đ^4k \mathrm{tr}\gamma^\mu \frac{\not{k}+m}{k^2-m^2} = 0$ by $\mathrm{tr}\gamma^\mu = 0$ and Lorentz symmetry, $\int^d 4k k^\mu f(k^2) = 0$. The one-point function for the photon also has to vanish by charge-conjugation symmetry (in fact any odd-point function of the photon does for the same reason; this is called Furry's theorem).

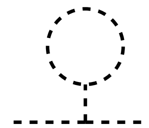
   More generally, a *tadpole diagram* – a diagram with a single field line coming out of it – represents a source for the field. When we developed our Feynman rules, we expanded around a minimum of the potential for the field, and this is why there is no one-point vertex in the Feynman rules. A tadpole diagram is saying that radiative effects are producing a shift in the minimum of the potential. The (quadratic part of the) action wants to change to $\int((\partial A)^2 - m_\gamma^2 A^2 + Aj)$. The equations of motion for the zero-momentum field tell us that the minimum is at $A = j/m_\gamma^2$. In the case of a massless field, the shift is arbitrarily large (in this linear approximation). This is the source of the IR divergence in the tadpole diagram as $m_\gamma \to 0$. In QED, this is moot because $j = 0$.

For the remainder of the problem, we consider $\phi^3$ theory with a (small) mass:

$$S = \int d^D x \left( \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3 \right).$$

(b) Notice that unlike $\phi^4$ theory (or QED), there is no symmetry that forbids a one-point function for the scalar. Why don't we lose generality by not adding a term linear in $\phi$ to the Lagrangian?

We can shift it away by a field redefinition, $\phi \to \phi - a$. It is convenient to choose $a$ to make the linear term vanish, since then the solution to the equations of motion has $\phi_0 = 0$.

(c) Now think about the following contribution to the scalar self-energy: 

Show that in the limit $m \to 0$ there is an IR divergence. By thinking about the significance for the scalar potential of this part of the diagram  explain the meaning of this divergence.

The object  is a one-point function for the scalar. As explained in the answer to the previous part of the problem, the presence of such a one-point function ($V_{\text{eff}} \ni v\phi$, with $v \propto g$) means we are doing perturbation theory about a configuration which is not a solution to the equations of motion at order $g$. The correct solution to the equations of motion is $\phi_0$ with $0 = m^2\phi_0 + v$ so $\phi_0 = -v/m^2$, which diverges when $m \to 0$. This is the origin of the IR divergence – the field theory is trying to find its minimum which, when $m \to 0$, is arbitrarily far away in field space.

3. **Bremsstrahlung.** [optional] Show that the number of photons per decade of wavenumber produced by the sudden acceleration of a charge is (in the relativistic limit $-q^2 \gg m^2$)

$$f_{IR}(q^2) = 2\frac{\alpha}{\pi} \ln\left(\frac{-q^2}{m^2}\right),$$

where $q_\mu = p'_\mu - p_\mu$ is the change of momentum and $m$ is the mass of the charge.

This is explained well on pages 177-182 of Peskin. The energy comes out to

$$U = \int đ^3 k \frac{2\alpha}{\pi} \ln\left(\frac{-q^2}{m^2}\right) = 2\int đ^3 k k N_k$$

where $N_k$ is the number density of photons of momentum $k$ of each polarization, and the RHS used the fact that each photon of momentum $k$ carries energy $k$. (The 2 comes from two polarizations for each momentum) Then the *number* of photons is

$$\mathcal{N} = \int \frac{dk}{k}\frac{\alpha}{\pi}\ln\left(\frac{-q^2}{m^2}\right) = \int d\log k \frac{\alpha}{\pi}\ln\left(\frac{-q^2}{m^2}\right)$$

and hence $2\frac{\alpha}{\pi}\ln\left(\frac{-q^2}{m^2}\right)$ is the total number of photons per decade of wavenumber. (Note that the integral over $k$ here actually diverges; this is an artifact of the approximation that the momentum change is instantaneous.)

4. **Soft photons.** [optional]

   Check that the contribution from a single virtual photon (eq 2.60 of the lecture notes, with $n = 2$) is

$$\frac{e^2}{2}\int đ^4 d\frac{-\mathbf{i}\eta_{\rho\sigma}}{k^2 - m_\gamma^2}\left(\frac{p'}{p'\cdot k} - \frac{p}{p\cdot k}\right)^\rho \left(\frac{p'}{-p'\cdot k} - \frac{p}{-p\cdot k}\right)^\sigma = -\frac{\alpha}{2\pi}f_{IR}(q^2)\ln\left(\frac{-q^2}{m_\gamma^2}\right) + \text{finite} \tag{1}$$

   where

$$f_{IR}(q^2) = \int_0^1 dx \frac{m_e^2 - q^2/2}{m_e^2 - x(1-x)q^2} - 1. \tag{2}$$

   [Hints: Wick rotate. Scale out the overall magnitude of $k$, $k^\mu = k\hat{k}^\mu$. Use Feynman parameters to combine $(p\cdot\hat{k})(p'\cdot\hat{k})$. ]

   Observe that this same integral appears in the cross section involving the emission of one real soft photon.

   See Peskin ...

5. **Scale invariance in QFT in $D = 0+0$, part 3.** [I got this problem from Frederik Denef.]

   We continue our study of QFT in $D = 0+0$ with two fields:

$$Z = \int dP_X dP_Y dX dY e^{-H/T}.$$

   Let's start by considering again

$$H = \frac{1}{2}P_X^2 + \frac{1}{2}P_Y^2 + V(X,Y), \qquad V(X,Y) = aX^4 + bY^8 \tag{3}$$

   for some nonzero constants $a, b$.

   A generic relevant deformation of (3) will flow to a Gaussian fixed point $V(X,Y) \sim X^2 + Y^2$ in the IR. Some other, more fine-tuned deformations will flow to other fixed points. For example, $\delta V(X,Y) = \epsilon Y^4$ will flow to $V(X,Y) = X^4 + Y^4$. But something more interesting happens for $\delta V(X,Y) = \epsilon X^2 Y^2$. This deformation is a relevant perturbation of (3) in the sense that $\delta V(\lambda^{1/4}X, \lambda^{1/8}Y) = \lambda^\kappa V(X,Y)$ with $\kappa = 3/4 < 1$. But it is not true that the model simply flows to a fixed point with $V \propto X^2Y^2$ in the IR. That's because the model with such a potential has a divergent partition function: $\int_{-\infty}^{\infty} dX \int_{-\infty}^{\infty} dY e^{-\epsilon X^2 Y^2/T} \propto \sqrt{\frac{T}{\epsilon}} \int \frac{dX}{|X|} = \infty$. We cannot throw away the higher-order terms because they regulate the large-$X$ and large-$Y$ behavior of the integral. Thus, in this model, the UV does not completely decouple from the IR. As a consequence, naive scaling arguments break down, and the partition function develops "anomalous" logarithmic dependence on $T$ for small $T$.

(a) Compute the partition function for the model (3) deformed by $\delta V(X,Y) = \epsilon X^2 Y^2$ analytically using Mathematica or some other symbolic software. This will give a horrible mess of hypergeometric functions. Expand it at small $T$ and you should find something of the form

$$Z = Z_0 T^c \log \frac{\Lambda}{T} \tag{4}$$

up to corrections suppressed by positive powers of $\sqrt{T/\Lambda}$. Find the constants $Z_0, c, \Lambda$. The over all normalization $Z_0$ does not mean anything in classical statistical mechanics.

Mathematica will tell you that the integral

$$Z_V = \int_{-\infty}^{\infty} dX dY e^{-(aX^4 + bY^8 + \epsilon X^2 Y^2)/T}$$

is

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Clear[a]; a1 = 4 Integrate[E^(-(a X^4 + b Y^8 + ϵ X^2 Y^2)/T), {X, 0, ∞}, {Y, 0, ∞}, Assumptions → {T > 0, ϵ > 0, a > 0, b > 0}]

1/(48 a^(7/4) Sqrt[b^3 T]) (192 a^(3/2) b^(11/8) T^(7/8) Gamma[9/8] Gamma[5/4] HypergeometricPFQ[{1/8, 1/8, 5/8}, {1/4, 1/2, 3/4}, ϵ^4/(64 a^2 b T)] -
  ϵ (6 a b^(9/8) T^(5/8) Gamma[3/8] Gamma[3/4] HypergeometricPFQ[{3/8, 3/8, 7/8}, {1/2, 3/4, 5/4}, ϵ^4/(64 a^2 b T)] +
    ϵ (-3 Sqrt[a] b^(7/8) T^(3/8) Gamma[5/8] Gamma[5/4] HypergeometricPFQ[{5/8, 5/8, 9/8}, {3/4, 5/4, 3/2}, ϵ^4/(64 a^2 b T)] +
      (b^5 T)^(1/8) ϵ Gamma[7/8] Gamma[7/4] HypergeometricPFQ[{7/8, 7/8, 11/8}, {5/4, 3/2, 7/4}, ϵ^4/(64 a^2 b T)])))
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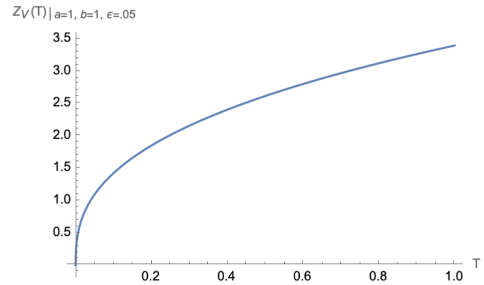
This function looks like:




The series expansion has a bit that goes like $\sqrt{T}\log T$ plus corrections of order $\sqrt{T}$, and a bit that goes like $Te^{\frac{\epsilon^4}{64a^2bT}}$. The latter is a very weird function. If it were $e^{-1/T}$ with a negative coefficient in the exponent, it would be easy to say that this is non-perturbatively small. With a positive but small coefficient (*i.e.* for small $\epsilon$) it is essentially indistinguishable from $T$, as long as $T > 0$. Therefore it is subleading. If you plot each of these bits individually, you can see that the former is the part that matters.

(b) Using (4), compute the dimensionless quantities $U/T$ and $C$. (Without the logarithmic dependence on $T$, these would be equal.) Check that in the strict limit $T \to 0$, you get the values for $U/T$ and $C$ that you would have guessed based on naive scaling arguments for $V \propto X^2Y^2$. Note that a logarithm varies more slowly than the $T^{1/2}$ corrections that we threw away.

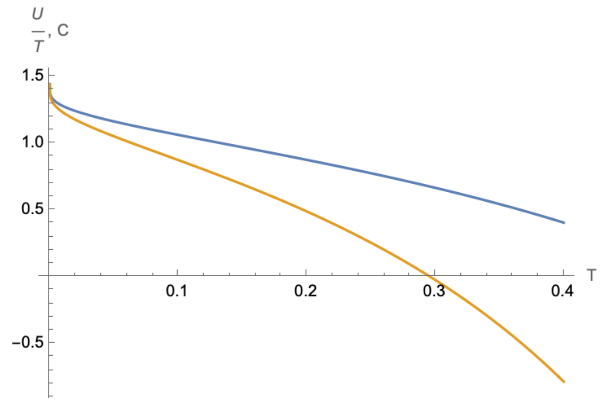
So $Z = Z_0 T^{1+\frac{1}{2}} \log T/\Lambda$ (don't forget the contribution from the two momentum integrals) and therefore

$$U/T = T\partial_T \log Z = \frac{3}{2} + \frac{1}{\log T/\Lambda} \tag{5}$$

while

$$C = \partial_T U = \frac{3}{2} + \frac{1}{\log T/\Lambda} - \frac{1}{\log^2 T/\Lambda}. \tag{6}$$

The naive answer is $Z \sim T^{1+1/2}$, using $Z_V \stackrel{?}{=} \int dXdYe^{-X^2Y^2\epsilon/T} = \sqrt{T/\epsilon}\int dxdye^{-x^2y^2}$ by scaling; this would work if the integral were actually well-defined without introducing some other scale. This gives $U/T = C = \frac{3}{2}$, and indeed both of the above functions do approach $\frac{3}{2}$ as $T \to 0$. The correct curves look like




(c) To what extent does the IR physics depend on the UV completion of the $V \propto X^2Y^2$ model? We could have started with $V = aX^8 + bY^8 + \epsilon X^2Y^2$ instead. This model would have different high-temperature physics. Redo part for this potential. You'll find an equally-horrendous, but different combination of hypergeometric functions. Which of the parameters $Z_0, c, \Lambda$ are the same?

Only $c$ is universal.

(d) The result of the previous part remains true for any other UV completion of the $V \propto X^2Y^2$ model, as long as $\delta V = \epsilon X^2Y^2$ remains a relevant deformation. In fact, we could equally well just take $V = \epsilon X^2Y^2$ and impose a hard cutoff on the $X$ and $Y$ integrals at some fixed values $|X| \leq X_0, |Y| \leq Y_0$ (this is like $V = X^n + Y^n$ with $n \to \infty$). Check that this again reduces to (4).

The answer is simpler:

$$Z_V^L \equiv \int_{-L}^{L} dX \int_{-L}^{L} dY e^{-\epsilon X^2Y^2/T} = 4L^2 \texttt{HypergeometricPFQ}\left[\left\{\frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{3}{2}, \frac{3}{2}\right\}, -\frac{L^4\epsilon}{T}\right].$$

This has the simpler low-temperature expansion:

$$Z_V^L \sim -\sqrt{\frac{\pi T}{\epsilon}} \log \frac{T}{\epsilon L^4 \gamma} + \mathcal{O}(T^{3/2}) + e^{-L^4\epsilon/T}\mathcal{O}(T^2)$$

where $\gamma$ is some irrelevant constant, and now the other term really is non-perturbatively small.

I just learned (from Rolando Ramirez-Camasco) about the Mathematica command `Asymptotic[]`, which does a better job than `Series[]` here.

(e) In view of this apparent universality of (4) at low $T$, it is desirable to have a way of deriving it without having to take the detour involving the horrendous hypergeometric functions. Here is one way. We use the hard cutoff $|X| \leq L, |Y| \leq L$, so that the position-space factor is

$$Z_V(T, L) = \int_{-L}^{L} dX \int_{-L}^{L} dY e^{-X^2Y^2/T} \tag{7}$$

where we've set $\epsilon = 1$ by a choice of temperature units. A rescaling of the integration variables $(X, Y) \to (T^{1/4}X, T^{1/4}Y)$ shows that $Z_V(T, L) = \sqrt{T}F(T^{-1/4}L)$ for some function $F$ of one variable. To find $F$, compute $L\partial_L Z_V$ directly from (5). By another suitable rescaling, show that $L\partial_L Z$ is finite and easily computable for $L^4/T \to \infty$. Infer from this the dependence on the cutoff $L$ in the regime $T \ll L^4$ and thus the function $F$ in this regime. This reproduces (4).

$$Z_V(T, L) = 4\int_0^L dX \int_0^L dY e^{-X^2Y^2/T} = \sqrt{T}F(T^{-1/4}L).$$

By the fundamental theorem of calculus,

$$L\partial_L Z_V = 4L \int_0^L dY e^{-L^2Y^2/T} \times 2$$

where the last factor of two comes from the place where the $L$ derivative hits the upper limit of the $Y$ integral. By scaling $y = L^2Y^2/T$ (so $dY = dy\sqrt{T}/L$) this is

$$L\partial_L Z_V = 8L\frac{\sqrt{T}}{L}\int_0^{L^2T^{-1/2}} dy e^{-y^2} = 8\sqrt{T}\left(\sqrt{\frac{\pi}{2}} + \mathcal{O}(e^{-L^4/T})\right).$$

Using $x\partial_x|_T = L\partial_L$, we have

$$x\partial x F(x) = T^{-1/2}L\partial_L Z_V = 4\sqrt{\pi} + \mathcal{O}(e^{-L^4/T}).$$

The solution of this ODE is $F(x) = c + 4\sqrt{\pi}\log x$, and therefore

$$Z_V(T, L) = \sqrt{\frac{T}{\epsilon}}(c + \sqrt{\pi}\log\frac{\epsilon L^4}{T}).$$

At the last step, I restored the $\epsilon$ by dimensional analysis. Since we don't care about the overall factor, we can get rid of the $\sqrt{\pi}$, and this is what we had above.

(f) We conclude that even when some kind of UV completion is required to give finite answers, the observable low-energy physics remains essentially independent of the UV completion. The infinite number of possible UV completions all flow in the IR to a partition function of the same form (4), with the details of the UV completion all lumped into a single scale parameter $\Lambda$. In fact, in the absence of other reference scales that can be used to fix a unit of temperature, the parameter $\Lambda$ does not really label physically distinct models, since we can always choose units with $\Lambda = 1$. Equivalently, only dimensionless quantities (and relations between them) are physically meaningful. Examples of such dimensionless quantities are $C$ and $u \equiv U/T$. Show that $C$ and $u$ obey a universal relation $C = f(u)$ with $f(u)$ independent of $T$ and $\Lambda$, and thus independent of the UV completion of the $X^2Y^2$ model. In the same spirit, show that the function $g(u)$ in the flow equation $T\partial_T u = g(u)$ is independent of the UV completion.

A brute force way to do this is just to compute them both from $Z = Z_0 T \log T/\Lambda$ and find the answers in (??) and (??). Letting $L \equiv \frac{1}{\log T/\Lambda}$, we have

$$u = \frac{3}{2} + L, C = \frac{3}{2} + L - L^2$$

so $L = u - \frac{3}{2}$ and

$$C = -u^2 + 3u - \frac{3}{2} \equiv f(u).$$

Similarly,

$$T\partial_T u = -\frac{1}{\log T/\Lambda} = -L^2 = -\left(u - \frac{3}{2}\right)^2 \equiv g(u).$$

(g) Show that on the other hand $f(u)$ and $g(u)$ *do* depend on the IR part of the potential, for example by comparing the IR potential $V = X^2Y^2$ considered above to another IR potential such as $V = X^6Y^6$.

If instead we used $\delta V = \epsilon X^6 Y^6$, we would find in part 5e instead

$$Z_V(T, L) = T^{1/6} F(T^{-1/12} L)$$

and

$$x\partial_x F(x) = T^{-\frac{1}{6}} L \partial_L Z_V = 8 \int_0^{L^2 T^{-\frac{1}{6}}} dy e^{-y^6} = 8\Gamma(7/6) + \mathcal{O}(e^{-L^6/\sqrt{T}}).$$

Therefore, in the limit $T \ll L^{12}$, the solution is

$$Z_V = T^{1/6}(c + 8\Gamma(7/6)\log(T^{\frac{-1}{12}} L))$$

and therefore

$$Z = Z_0 T^{1+\frac{1}{6}} \log T/\Lambda$$

and

$$u = U/T = T\partial_T \log Z = \frac{7}{6} + \frac{1}{\log T/\Lambda} = \frac{2}{3} + L \tag{8}$$

while

$$C = \partial_T U = \frac{7}{6} + \frac{1}{\log T/\Lambda} - \frac{1}{\log^2 T/\Lambda} = \frac{7}{6} + L - L^2. \tag{9}$$

These satisfy $L = u - \frac{7}{6}$, so

$$C = u - \left(u - \frac{7}{6}\right)^2 = f(u)$$

and $T\partial_T u = -L^2 = -\left(u - \frac{7}{6}\right)^2 = g(u)$ are indeed different.