University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2025
Assignment 5

Due 11:59pm Tuesday, February 11, 2025

---

1. **Brain-warmer.**

   Prove the Gordon identities

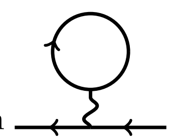
$$\bar{u}_2 \left(q^\nu \sigma_{\mu\nu}\right) u_1 = \mathbf{i}\bar{u}_2 \left((p_1 + p_2)_\mu - (m_1 + m_2)\gamma_\mu\right) u_1$$

   and

$$\bar{u}_2 \left((p_1 + p_2)^\nu \sigma_{\mu\nu}\right) u_1 = \mathbf{i}\bar{u}_2 \left((p_2 - p_1)_\mu - (m_2 - m_1)\gamma_\mu\right) u_1$$

   where $q \equiv p_2 - p_1$ and $\not{p}_1 u_1 = m_1 u_1, \bar{u}_2 \not{p}_2 = m_2 \bar{u}_2$, using the definitions and the Clifford algebra.

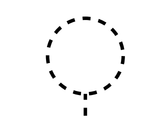
2. **Tadpole diagrams.**

   (a) Why don't we worry about the following diagram  as a correction to the electron self-energy in QED?

   For the remainder of the problem, we consider ϕ^3 theory with a (small) mass:

$$S = \int d^D x \left(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3 \right).$$

   (b) Notice that unlike ϕ^4 theory (or QED), there is no symmetry that forbids a one-point function for the scalar. Why don't we lose generality by not adding a term linear in ϕ to the Lagrangian?

   (c) Now think about the following contribution to the scalar self-energy: 

   Show that in the limit $m \to 0$ there is an IR divergence. By thinking about the significance for the scalar potential of this part of the diagram  explain the meaning of this divergence.

3. **Bremsstrahlung.** [optional] Show that the number of photons per decade of wavenumber produced by the sudden acceleration of a charge is (in the relativistic limit $-q^2 \gg m^2$)

$$f_{IR}(q^2) = 2\frac{\alpha}{\pi} \ln \left(\frac{-q^2}{m^2} \right),$$

where $q_\mu = p'_\mu - p_\mu$ is the change of momentum and m is the mass of the charge.

4. **Soft photons.** [optional]

Check that the contribution from a single virtual photon (eq 2.60 of the lecture notes, with $n = 2$) is

$$\frac{e^2}{2} \int đ^4 d \frac{-\mathbf{i}\eta_{\rho\sigma}}{k^2 - m_\gamma^2} \left(\frac{p'}{p' \cdot k} - \frac{p}{p \cdot k} \right)^\rho \left(\frac{p'}{-p' \cdot k} - \frac{p}{-p \cdot k} \right)^\sigma = -\frac{\alpha}{2\pi} f_{IR}(q^2) \ln \left(\frac{-q^2}{m_\gamma^2} \right) + \text{finite} \tag{1}$$

where

$$f_{IR}(q^2) = \int_0^1 dx \frac{m_e^2 - q^2/2}{m_e^2 - x(1-x)q^2} - 1. \tag{2}$$

[Hints: Wick rotate. Scale out the overall magnitude of k, $k^\mu = k\hat{k}^\mu$. Use Feynman parameters to combine $(p \cdot \hat{k})(p' \cdot \hat{k})$.]

Observe that this same integral appears in the cross section involving the emission of one real soft photon.

5. **Scale invariance in QFT in $D = 0 + 0$, part 3.** [I got this problem from Frederik Denef.]

We continue our study of QFT in $D = 0 + 0$ with two fields:

$$Z = \int dP_X dP_Y dX dY e^{-H/T}.$$

Let's start by considering again

$$H = \frac{1}{2} P_X^2 + \frac{1}{2} P_Y^2 + V(X, Y), \qquad V(X, Y) = aX^4 + bY^8 \tag{3}$$

for some nonzero constants a, b.

A generic relevant deformation of (3) will flow to a Gaussian fixed point $V(X, Y) \sim X^2 + Y^2$ in the IR. Some other, more fine-tuned deformations will flow to other fixed points. For example, $\delta V(X, Y) = \epsilon Y^4$ will flow to $V(X, Y) = X^4 + Y^4$. But something more interesting happens for $\delta V(X, Y) = \epsilon X^2 Y^2$. This deformation is a relevant perturbation of (3) in the sense that $\delta V(\lambda^{1/4} X, \lambda^{1/8} Y) = \lambda^\kappa V(X, Y)$ with $\kappa = 3/4 < 1$. But it is not true that the model simply flows to a fixed point with $V \propto X^2Y^2$ in the IR. That's because the model with such a potential has a divergent partition function: $\int_{-\infty}^{\infty} dX \int_{-\infty}^{\infty} dY e^{-\epsilon X^2Y^2/T} \propto \sqrt{\frac{T}{\epsilon}} \int \frac{dX}{|X|} = \infty$. We cannot throw away the higher-order terms because they regulate the large-X and large-Y behavior of the integral. Thus, in this model, the UV does not completely decouple from the IR. As a consequence, naive scaling arguments break down, and the partition function develops "anomalous" logarithmic dependence on T for small T.

(a) Compute the partition function for the model (3) deformed by $\delta V(X,Y) = \epsilon X^2Y^2$ analytically using Mathematica or some other symbolic software. This will give a horrible mess of hypergeometric functions. Expand it at small T and you should find something of the form

$$Z = Z_0 T^c \log \frac{\Lambda}{T} \tag{4}$$

up to corrections suppressed by positive powers of $\sqrt{T/\Lambda}$. Find the constants Z_0, c, Λ. The over all normalization Z_0 does not mean anything in classical statistical mechanics.

(b) Using (4), compute the dimensionless quantities U/T and C. (Without the logarithmic dependence on T, these would be equal.) Check that in the strict limit $T \to 0$, you get the values for U/T and C that you would have guessed based on naive scaling arguments for $V \propto X^2Y^2$. Note that a logarithm varies more slowly than the $T^{1/2}$ corrections that we threw away.

(c) To what extent does the IR physics depend on the UV completion of the $V \propto X^2Y^2$ model? We could have started with $V = aX^8 + bY^8 + \epsilon X^2Y^2$ instead. This model would have different high-temperature physics. Redo part for this potential. You'll find an equally-horrendous, but different combination of hypergeometric functions. Which of the parameters Z_0, c, Λ are the same?

(d) The result of the previous part remains true for any other UV completion of the $V \propto X^2Y^2$ model, as long as $\delta V = \epsilon X^2Y^2$ remains a relevant deformation. In fact, we could equally well just take $V = \epsilon X^2Y^2$ and impose a hard cutoff on the X and Y integrals at some fixed values $|X| \leq X_0, |Y| \leq Y_0$ (this is like $V = X^n + Y^n$ with $n \to \infty$). Check that this again reduces to (4).

(e) In view of this apparent universality of (4) at low T, it is desirable to have a way of deriving it without having to take the detour involving the horrendous hypergeometric functions. Here is one way. We use the hard cutoff $|X| \leq L, |Y| \leq L$, so that the position-space factor is

$$Z_V(T,L) = \int_{-L}^{L} dX \int_{-L}^{L} dY e^{-X^2Y^2/T} \tag{5}$$

where we've set $\epsilon = 1$ by a choice of temperature units. A rescaling of the integration variables $(X,Y) \to (T^{1/4}X, T^{1/4}Y)$ shows that $Z_V(T,L) = \sqrt{T}F(T^{-1/4}L)$ for some function F of one variable. To find F, compute $L\partial_L Z_V$ directly from (5). By another suitable rescaling, show that $L\partial_L Z$ is finite and easily computable for $L^4/T \to \infty$. Infer from this the dependence on the cutoff L in the regime $T \ll L^4$ and thus the function F in this regime. This reproduces (4).

(f) We conclude that even when some kind of UV completion is required to give finite answers, the observable low-energy physics remains essentially independent of the UV completion. The infinite number of possible UV completions all flow in the IR to a partition function of the same form (4), with the details of the UV completion all lumped into a single scale parameter Λ. In fact, in the absence of other reference scales that can be used to fix a unit of temperature, the parameter Λ does not really label physically distinct models, since we can always choose units with $\Lambda = 1$. Equivalently, only dimensionless quantities (and relations between them) are physically meaningful. Examples of such dimensionless quantities are C and $u \equiv U/T$. Show that C and u obey a universal relation $C = f(u)$ with $f(u)$ independent of T and Λ, and thus independent of the UV completion of the X^2Y^2 model. In the same spirit, show that the function $g(u)$ in the flow equation $T\partial_T u = g(u)$ is independent of the UV completion.

(g) Show that on the other hand $f(u)$ and $g(u)$ *do* depend on the IR part of the potential, for example by comparing the IR potential $V = X^2Y^2$ considered above to another IR potential such as $V = X^6Y^6$.