University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2025
Assignment 6 – Solutions

Due 11:59pm Tuesday, February 18, 2025

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1. **Brain-warmer.** Check that $(\Delta_T)^\mu_\rho \equiv \delta^\mu_\rho - \frac{q^\mu q_\rho}{q^2}$ is a projector onto momenta transverse to q^ρ. This requires showing both that $\Delta q = 0$ and that $\Delta^2 = \Delta$.

2. **A better formula for the superficial degree of divergence.** [Thanks to Haoran Sun for suggesting this formula.]

   Starting from the definition of an (amputated!) amplitude $\mathcal{A}$ from a connected Feynman diagram, show that its superficial degree of divergence is

$$D_{\mathcal{A}} = D - \sum_{\{g\}}[g]V_g(\mathcal{A}) - \sum_{\{f\}} E_f(\mathcal{A})[f] \tag{1}$$

   where $\{g\}$ is the set of coupling constants and $\{f\}$ is the set of fields, $V_g(\mathcal{A})$ is the number of vertices of the coupling g in the diagram $\mathcal{A}$, and $E_f(\mathcal{A})$ is the number of external f fields. For example, for the Yukawa theory you studied on a previous homework, this formula reduces to

$$D_{\mathcal{A}} = D - [g]V_g - [y]V_y - B_E[\phi] - F_E[\phi] \tag{2}$$

   where $B_E \equiv E_\phi$ is the number of external scalars and $F_E \equiv E_\psi$ is the number of external fermions in the diagram. If you prefer, for definiteness, you could just show the formula for this case.

   The key idea is that, schematically,

$$\mathcal{A}\delta^D(\sum p) \sim \prod^{\sum_f E_f} \left(\frac{1}{D_f}\int d^D x_i e^{\mathbf{i}p_i x_i}\right) \langle 0| \prod_f f^{E_f} \prod_g \left(gG_g(f)\right)^{V_g} |0\rangle \tag{3}$$

   where $D_f = \int d^D x \langle f(x) f(0)\rangle e^{\mathbf{i}px}$ is the momentum space propagator for f (don't forget that $\mathcal{A}$ is amputated!), and $G_g(f)$ is the interaction term whose coupling constant is g. This seems a bit too abstract, so let's do it for the case of the Yukawa theory and you'll see that the idea is general. In that case the formula above becomes

$$\mathcal{A}\delta^D(\sum p) \sim \prod_i^{B_E} p_i^2 \prod_i^{F_E} \not{p}_i \prod^{F_E+B_E} \int \left(d^D x_i e^{\mathbf{i}p_i x_i}\right) \langle 0| \phi^{B_E}\psi^{F_E} \left(y\int d^D y \phi\bar{\psi}\psi\right)^{V_y} \left(g\int d^D y \phi^4(y)\right)^{V_g} |0\rangle \tag{4}$$

It's important that $\mathcal{A}$ corresponds to a particular connected contraction of the object on the RHS. But we're just here to count the mass dimension (when we set the couplings to one). Thus

$$D_{\mathcal{A}} - D = 2B_E + F_E - DF_E - DB_E + B_E[\phi] + F_E[\psi] + V_y\left(-D + [\phi] + 2[\psi]\right) + V_g\left(-D + 4[\phi]\right). \tag{5}$$

Using the fact that

$$[y] = -D + [\phi] + 2[\psi], [g] = -D + 4[\phi], \tag{6}$$

and

$$[\phi] = \frac{D-2}{2}, [\psi] = \frac{D-1}{2} \tag{7}$$

this gives the promised formula.

3. **Symmetry is attractive.** Consider a field theory in $D = 3+1$ with two scalar fields with the same mass which interact via the interaction

$$V = -\frac{g}{4!}\left(\phi_1^4 + \phi_2^4\right) - \frac{2\lambda}{4!}\phi_1^2\phi_2^2.$$

 (a) Show that when $\lambda = g$ the model possesses an $\mathsf{O}(2)$ symmetry.

 At this special point, the potential is $(\phi_1^2 + \phi_2^2)^2$, which depends only on the distance from the origin of the field space.

 (b) Will you need a counterterm of the form $\phi_1\phi_2$ or $\phi_1\Box\phi_2$ (for general g, λ)? If not, why not?

 A very important point: such terms can't be generated because they violate the $\mathbb{Z}_2$ symmetry that takes $(\phi_1, \phi_2) \to (-\phi_1, \phi_2)$. In general, radiative effects (*i.e.* loops) will not violate symmetries of the bare action. Exceptions to this statement are called *anomalies*; this only happens when no regulator preserves the symmetry in question.

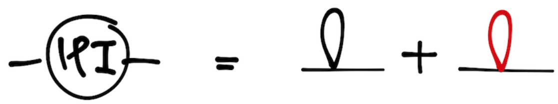
 (c) Renormalize the theory to one loop order by regularizing (for example with a euclidean momentum cutoff or Pauli Villars), adding the necessary counterterms, and imposing a renormalization condition on the propagators (consider the case where the scalars are both massless) and $2 \to 2$ scattering amplitudes at some values of the kinematical variables s_0, t_0, u_0. Feel free to re-use our results from ϕ^4 theory where appropriate.

 I'll use a hard euclidean momentum cutoff since then we can reuse our results from ϕ^4 theory. To save typing let me define $L(x) \equiv \frac{1}{32\pi^2}\log x$. Every loop integral we will encounter is the same as in the pure massless ϕ^4 theory that we did in lecture.

The symmetry that interchanges $\phi_1 \leftrightarrow \phi_2$ guarantees that their self-couplings g (and the masses) stay equal (using the same principle as above). This means we have only three counterterms to determine altogether: δ_{m^2} and two four-point counterterms $(\delta_g, \delta_\lambda)$. That is, we have to impose two renormalization conditions on the four-point functions.

First an annoying point: with the given normalization, the 1122 vertex is actually $-\mathbf{i}\lambda/3$.

The self-energy for ϕ_1 is

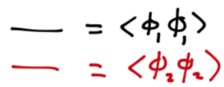
$-\mathbf{i}\Sigma(p^2) =$ 

$+.. = -\mathbf{i}(g+\lambda/3)c\Lambda^2 + \mathcal{O}(g,\lambda)^2$

where c is a numerical constant that I can't remember right now and which we don't need. To put the pole at $p^2 = m_P^2 = 0$, we need the bare mass to be

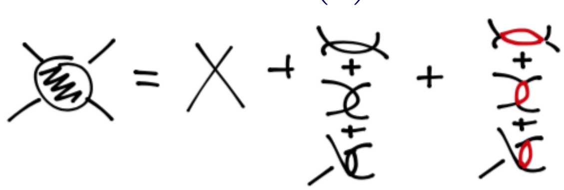
$$m^2(\Lambda) = -\Sigma(p^2 = 0) = (g + \lambda/4)c\Lambda^2.$$

As in ϕ^4 theory, there is no wavefunction renormalization at one loop because Σ is independent of p^2.

There are three different $2 \to 2$ scattering processes to consider: $11 \to 11, 11 \to 22, 12 \to 12$. (The corrections to $22 \to 22$ are the same as those for $11 \to 11$, and similiarly $22 \to 11$ is the same as $11 \to 22$, by the exchange symmetry.) Then using the notation 

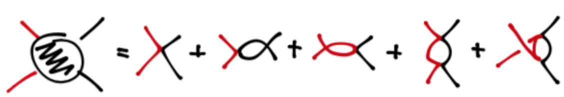
we have

$$\mathcal{M}_{11\leftarrow 11} = -g + (g^2 + \left(\frac{\lambda}{3}\right)^2)(L(s/\Lambda^2) + L(t/\Lambda^2) + L(u/\Lambda^2)) + \delta_g \quad (8)$$


(9)

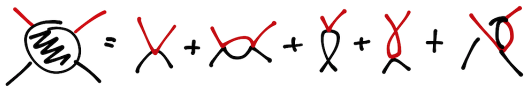
The λ^2 term involves ϕ_2 running in the loop. (Note that I am writing $\mathbf{i}\mathcal{M} = -\mathbf{i}g + (-\mathbf{i}g)^2$... and dividing the BHS by $\mathbf{i}$.) Beware the symmetry factor of $\frac{1}{2}$ in each loop diagram.

$$\mathcal{M}_{22\leftarrow 11} = -\frac{\lambda}{3} + \frac{\lambda}{3}g2L(s/\Lambda^2) + \left(\frac{\lambda}{3}\right)^2 \left(2L(t/\Lambda^2) + 2L(u/\Lambda^2)\right) + \delta_\lambda \quad (10)$$


(11)

where the 2 in the s-channel term is from the fact that either ϕ_1 or ϕ_2 can run in the loop. The last two diagrams have a different symmetry factor from the others, since we can't exchange the two propagators in the loop – so they get an extra factor of 2.

$$\mathcal{M}_{12\leftarrow 12} = -\frac{\lambda}{3} + \left(\frac{\lambda}{3}\right)^2 \left(2L(s/\Lambda^2) + 2L(u/\Lambda^2)\right) + 2\frac{\lambda}{3} g L(t/\Lambda^2) + \delta_\lambda \quad (12)$$



(13)

Using the renormalization conditions $\mathcal{M}_{11\leftarrow 11}(s_0 = t_0 = u_0) = -g_P$ and $\mathcal{M}_{22\leftarrow 11}(s_0 = t_0 = u_0) = -\frac{\lambda_P}{3}$ we find

$$\lambda(\Lambda) \equiv \lambda + \delta_\lambda = \lambda_P + \lambda_P 2 g_P L + 4\frac{\lambda_P^2}{3} L + \mathcal{O}(\lambda_P, g_P)^2 \quad (14)$$

$$g(\Lambda) \equiv g + \delta_g = g_P + \left(g_P^2 + \left(\frac{\lambda_P}{3}\right)^2\right) 3L + \mathcal{O}(\lambda_P, g_P)^2 \quad (15)$$

where $L \equiv L(s_0/\Lambda^2)$. We've solved for the couplings perturbatively, to second order in both, which means we ignored the difference between *e.g.* g and g_P in the quadratic term, as we must. From now on I will drop the P subscripts on the physical coupling.

Notice that we would get the same answer if we defined λ_P by fixing a value of $\mathcal{M}_{12\leftarrow 12}$ instead. This is because of crossing symmetry.

(d) Consider the limit of low energies, *i.e.* when $s_0, t_0, u_0 \ll \Lambda^2$ where Λ is the cutoff scale. Tune the location of the poles in both propagators to $p^2 = 0$. Show that the coupling goes to the $\mathsf{O}(2)$-symmetric value if it starts nearby (nearby means $\lambda/g < 3$). (That is, show that at fixed physical coupling, the ratio of bare couplings $\lambda/g \to 1$ as we take the cutoff to infinity.) A nice way to organize this is by computing the beta function for the coupling λ/g.

A nice trick for doing this is to compute the beta functions.

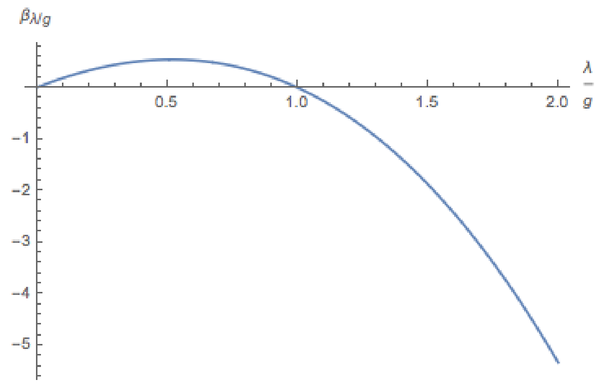
$$\beta_g \equiv 32\pi^2 \Lambda^2 \partial_{\Lambda^2} g(\Lambda) = 3\left(g^2 + \left(\frac{\lambda}{3}\right)^2\right), \quad \beta_\lambda \equiv 32\pi^2 \Lambda^2 \partial_{\Lambda^2} \lambda(\Lambda) = \left(2\lambda g + 4\frac{\lambda^2}{3}\right)$$

where I've pulled out a factor of $32\pi^2$ in the definition of β for convenience – it only affects how fast the flow happens. A useful check is that if we set $\lambda = 0$, we reproduce the beta function for ϕ^4 theory, $\beta_g = +3g^2$ (the 3 comes from the 3 different channels).

To look at the relative flow of g and λ let's compute

$$\beta_{\lambda/g} \equiv 8\pi^2\Lambda^2\partial_{\Lambda^2}\frac{\lambda}{g} = \frac{1}{g^2}\left(g\beta_\lambda - \lambda\beta_g\right) \propto \left(-\frac{\lambda^3}{3} - \frac{5}{3}g\lambda^2 + 2g^2\lambda\right) = \frac{1}{3}\lambda(\lambda - g)(\lambda + 6g).$$

This looks like this:



with the convention I'm using, positive β means that as we increase Λ, the coupling decreases. This means that the couplings approach the point $g = \lambda$ as $\Lambda \to \infty$ fixing g_P, λ_P. This is the case as long as we start with $\lambda/g < 3$.

4. **Yes, please, gauge invariance.** Verify for yourself that the one-loop vacuum polarization amplitude in QED (when computed using either the improved Pauli-Villars regulator or dim reg) satisfies the Ward identity, *i.e.* is proportional to $q^\mu q^\nu - \eta^{\mu\nu}q^2$. It's up to you how much of this to hand in.

   The calculation is done using dim reg on pages 251-252 of Peskin and using PV in Zee (2d ed) pages 202-204.

5. **Soft gravitons?** [optional] Photons are massless, and this means that the cross sections we measure actually include soft ones that we don't detect. If we don't include them we get IR-divergent nonsense.

   Gravitons are also massless. Do we need to worry about them in the same way? Here we'll sketch some hints for how to think about this question.

   (a) Consider the action

$$S_0[h_{\mu\nu}] = \int d^4x \frac{1}{2}h_{\mu\nu}\Box h^{\mu\nu}.$$

   This is a kinetic term for (too many polarizations of a) two-index symmetric-tensor field $h_{\mu\nu} = h_{\nu\mu}$ (which we'll think of as a small fluctuation of the metric about flat space: $g_{\mu\nu} \simeq \eta_{\mu\nu} + h_{\mu\nu}$, and this is where the coupling below comes from). Like with the photon, we'll rely on the couplings to matter to keep unphysical polarizations from being made. Write the propagator for h. We still raise and lower indices with $\eta_{\mu\nu}$. [1]

The propagator is simply the inverse of the kinetic term. After the gauge fixing (implicit in the expression I gave) it is indeed invertible, just like in Maxwell theory. The graviton propagator you'll find on Wikipedia is the propagator for $h_{\mu\nu}$, rather than $\bar{h}_{\mu\nu}$.

(b) Couple the graviton to the electron field via

$$S_G = \int d^4x \; G h^{\mu\nu} T_{\mu\nu}$$
$$T_{\mu\nu} \equiv \bar{\psi} \left(\gamma_\mu \partial_\nu + \gamma_\nu \partial_\mu\right) \psi. \tag{16}$$

What are the engineering dimensions of the coupling constant G? What is the new Feynman rule?

$G = \sqrt{G_N}$ has dimensions of one over mass. This factor pops out of $L \sim \frac{1}{G_N}\sqrt{g}R + T_{\mu\nu}h^{\mu\nu}$ upon rescaling h to give it canonical kinetic terms.

(c) Draw a (tree level) Feynman diagram that describes the creation of gravitational radiation from an electron as a result of its acceleration from the absorption of a photon ($e\gamma \to eh$). Evaluate it if you dare. Estimate or calculate the cross section (hint: use dimensional analysis).

The two diagrams that contribute at tree level are similar to those appearing in Compton scattering. The external graviton in the final state comes with a polarization tensor $\epsilon^\star_{\mu\nu}$. Since we don't measure the polarization of the soft graviton, we want the polarization-summed cross-section. As for photons, the polarization sum (there are two polarizations in $D = 4$)

$$\sum_r \epsilon_r^{\mu\nu}(q)\epsilon_r^{\rho\sigma}(q) = \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} + \text{terms with } q^\mu \text{ or } q^\nu$$

[1] A warning: I've done two misdeeds in the statement of this problem. First, the Einstein-Hilbert term is $\int d^4x \frac{1}{8\pi G_N}\sqrt{g}R = \int d^4x \frac{1}{8\pi G_N}(\partial h)^2 + \ldots$ – it has a factor of G_N in front of it. R has units of $\frac{1}{\text{length}^2}$, and g is dimensionless, so G_N has units of length^2 – it is $8\pi G_N = \frac{1}{M_{\text{Pl}}^2}$, where M_{Pl} is the Planck mass. I've absorbed a factor of $\sqrt{G_N}$ into h so that the coefficient of the kinetic term is unity. Second, the $(\partial h)^2$ here involves various index contractions, which I haven't written. Some gauge fixing (de Donder gauge) is required to arrive at the simple expression I wrote above, and one more thing – the $h_{\mu\nu}$ I've written is actually the 'trace-reversed' graviton field

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}$$

where $h \equiv \eta^{\mu\nu}h_{\mu\nu}$ is the trace. (I didn't write the bar.) For the details of this, which are not needed for this problem, see chapter 10 of my GR notes.

takes the same form as the numerator of the propagator, up to ambiguous terms that vanish because of the Ward identity for the gauge invariance described below.

The amplitude has a single factor of G, so the probability goes like G^2 which has dimensions of length-squared, which is already the right dimensions for a cross section. Apparently, for energies large compared to the electron mass, this cross section is constant in energy.

(d) Now the main event: study the one-loop diagram by which the graviton corrects the QED vertex. Is it IR divergent? If not, why not?

There are extra powers of the momentum in the numerator from the derivative coupling. This paper shows that this is not enough to prevent an IR divergence. So indeed, if we wish to include the (very small!) radiative corrections from gravitons, we must study inclusive amplitudes that allow for soft gravitons.

(e) If you get stuck on the previous part, replace the graviton field by a massless scalar $\pi(x)$. Compare the case of derivative coupling $\lambda \partial_\mu \pi \bar{\psi} \gamma^\mu \psi$ with the more direct Yukawa coupling $y\pi\bar{\psi}\psi$. [Warning: though this example has some similarities with the graviton case, the conclusion is different.]

In this case, the extra powers of the momentum in the numerator from the derivative coupling do prevent an IR divergence.

(f) Quite a bit about the form of the coupling of gravity to matter is determined by the demand of coordinate invariance. This plays a role like gauge invariance in QED. Acting on the small fluctuation, the transformation is

$$h_{\mu\nu}(x) \to h_{\mu\nu}(x) + \partial_\mu \lambda_\nu(x) + \partial_\nu \lambda_\mu(x).$$

What condition does the invariance under this (infinitesimal) transformation impose on the object $T_{\mu\nu}$ appearing in (3).

The variation of the action is

$$\delta S = \int d^4x \left(\partial_\mu \lambda_\nu(x) + \partial_\nu \lambda_\mu(x)\right) T^{\mu\nu} \overset{\text{IBP}}{=} -2 \int d^4x \lambda_\mu \partial_\nu T^{\mu\nu}$$

which vanishes if $\partial_\nu T^{\mu\nu} = 0$, *i.e.* if $T^{\mu\nu}$ is a conserved stress tensor.

6. **Equivalent photon approximation.** [optional] Consider a process in which very high-energy electrons scatter off a target. At leading order in α, the electron line is connected to the rest of the diagram by a single photon propagator. If the initial and final energies of the electron are E and E', the photon will carry momentum q with $q^2 = -2EE'(1-\cos\theta)$ (ignoring the electron mass $m \ll E$). In

the limit of forward scattering ($\theta \to 0$), we have $q^2 \to 0$, so the photon approaches its mass shell. In this problem, we ask: To what extent can we treat it as a real photon?

(a) The matrix element for the scattering process can be written as

$$\mathcal{M} = -\mathbf{i}e\bar{u}(p')\gamma^\mu u(p)\frac{-\mathbf{i}\eta_{\mu\nu}}{q^2}\hat{\mathcal{M}}^\nu(q)$$

where $\hat{\mathcal{M}}^\nu$ represents the coupling of the virtual photon to the target. Let $q = (q^0, \vec{q})$ and define $\tilde{q} = (q^0, -\vec{q})$. The contribution to the amplitude from the electron line can be parametrized as

$$\bar{u}(p')\gamma^\mu u(p) = Aq^\mu + B\tilde{q}^\mu + C\epsilon_1^\mu + D\epsilon_2^\mu$$

where ϵ_α are unit vectors transverse to $\vec{q}$. Show that B is at most of order θ^2 (dot it with q), so we can ignore it at leading order in an expansion about forward scattering. Why do we not care about the coefficient A?

Dotting with q, the terms with ϵ vanish since the polarizations are transverse and the Ward identity gives $0 = Aq^2 + Bq_\mu\bar{q}^\mu$, but $q^2 = -2EE'(1-\cos\theta) \sim \theta^2$ when $\theta \ll 1$. Since $q_\mu\bar{q}^\mu$ is order 1, B must be order θ^2. The term with A drops out when we contract this with $\hat{\mathcal{M}}_\mu$, again by the Ward identity.

(b) Working in the frame with $p = (E, 0, 0, E)$, compute

$$\bar{u}(p')\gamma \cdot \epsilon_\alpha u(p)$$

explicitly using massless electrons, where $\bar{u}$ and u are spinors of definite helicity, and $\epsilon_{\alpha=\parallel,\perp}$ are unit vectors parallel and perpendicular to the plane of scattering. Keep only terms through order θ. Note that for $\epsilon_\parallel$, the (small) $\hat{3}$ component matters.

Choose definite helicity states, say

$$u^+(p) = \sqrt{2E}\begin{pmatrix}0 & 0 & 1 & 0\end{pmatrix}^t, u^-(p) = \sqrt{2E}\begin{pmatrix}0 & 1 & 0 & 0\end{pmatrix}^t$$

for $\vec{p} \propto \hat{z}$, $i.e. p^\mu = (E, 0, 0, E)^\mu$. The two definite-helicity spinors for momentum $p' = (E', 0, E'sin\theta, E'\cos\theta)$ are related by a spinor rotation by angle θ, so

$$u^+(p') = \sqrt{2E'}\begin{pmatrix}0 & 0 & \cos\theta/2 & \sin\theta/2\end{pmatrix}^t \simeq \sqrt{2E'}\begin{pmatrix}0 & 0 & 1 & \theta/2\end{pmatrix}^t,$$

$$u^-(p') = \sqrt{2E'}\begin{pmatrix}-\sin\theta/2 & \cos\theta/2 & 0 & 0\end{pmatrix}^t \simeq \sqrt{2E'}\begin{pmatrix}-\theta/2 & 1 & 0 & 0\end{pmatrix}^t.$$

We must also find expressions for the polarization vectors:

$$\epsilon_\perp = (0,0,1,0), \epsilon_\parallel \propto (0, p'\cos\theta - p, 0, p'\sin\theta) \propto (0,1,0,\frac{E'}{E'-E}\theta).$$

Then

$$\bar{u}'_\pm \gamma \cdot \epsilon_\parallel u_\pm = \pm \mathbf{i}\sqrt{EE'}\theta, \bar{u}'_\pm \gamma \cdot \epsilon_\parallel u_\pm = -\sqrt{EE'}\frac{E+E'}{E-E'}\theta,$$

and $\bar{u}'_\pm \gamma \cdot \epsilon_\alpha u_\mp = 0$. The key conclusion is that all the nonzero entries in $\bar{u}_\pm(p')\gamma \cdot \epsilon_\alpha u_\pm(p)$ are order θ.

(c) Now write the expression for the electron scattering cross section, in terms of $|\hat{\mathcal{M}}^\mu|^2$ and the integral over phase space of the target. This expression must be integrated over the final electron momentum $\vec{p}'$. The integral over $p^{3'}$ is an integral over the energy loss of the electron. Show that the integral over $p'_\perp$ diverges logarithmically as $p'_\perp$ or $\theta \to 0$.

We find $|\mathcal{M}|^2 \propto \theta^2$. Then

$$\sigma \propto \int_0 d\theta \sin\theta \frac{|\mathcal{M}|^2}{q^4} \sim \int_0 d\theta \frac{\theta^3}{\theta^4}$$

is log divergent.

(d) The divergence as $\theta \to 0$ is regulated by the electron mass (which we've ignored above). Show that reintroducing the electron mass in the expression

$$q^2 = -2(EE' - pp'\cos\theta) + 2m^2$$

cuts off the divergence and gives a factor of $\log(s/m^2)$ in its place.

Just replace the denominator q^4 with this regulated expression.

(e) Assembling all the factors, and assuming that the target cross sections are independent of photon polarization, show that the largest part of the electron-target cross section is given by considering the electron to be the source of a beam of real photons with energy distribution given by

$$N_\gamma(x)dx = \frac{dx}{x}\frac{\alpha}{2\pi}(1 + (1-x)^2)\log\frac{s}{m^2}$$

where $x \equiv E_\gamma/E$. This is the Weizsäcker-Williams equivalent photon approximation. It is a precursor to the theory of jets and partons in QCD.