University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2025
Assignment 6

Due 11:59pm Tuesday, February 18, 2025

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1. **Brain-warmer.** Check that $(\Delta_T)^\mu_\rho \equiv \delta^\mu_\rho - \frac{q^\mu q_\rho}{q^2}$ is a projector onto momenta transverse to q^ρ. This requires showing both that $\Delta q = 0$ and that $\Delta^2 = \Delta$.

2. **A better formula for the superficial degree of divergence.** [Thanks to Haoran Sun for suggesting this formula.]

   Starting from the definition of an (amputated!) amplitude $\mathcal{A}$ from a connected Feynman diagram, show that its superficial degree of divergence is

$$D_\mathcal{A} = D - \sum_{\{g\}} [g] V_g(\mathcal{A}) - \sum_{\{f\}} E_f(\mathcal{A})[f] \tag{1}$$

   where $\{g\}$ is the set of coupling constants and $\{f\}$ is the set of fields, $V_g(\mathcal{A})$ is the number of vertices of the coupling g in the diagram $\mathcal{A}$, and $E_f(\mathcal{A})$ is the number of external f fields. For example, for the Yukawa theory you studied on a previous homework, this formula reduces to

$$D_\mathcal{A} = D - [g] V_g - [y] V_y - B_E[\phi] - F_E[\phi] \tag{2}$$

   where $B_E \equiv E_\phi$ is the number of external scalars and $F_E \equiv E_\psi$ is the number of external fermions in the diagram. If you prefer, for definiteness, you could just show the formula for this case.

3. **Symmetry is attractive.** Consider a field theory in $D = 3 + 1$ with two scalar fields with the same mass which interact via the interaction

$$V = -\frac{g}{4!}\left(\phi_1^4 + \phi_2^4\right) - \frac{2\lambda}{4!}\phi_1^2\phi_2^2.$$

   (a) Show that when $\lambda = g$ the model possesses an $\mathsf{O}(2)$ symmetry.

   (b) Will you need a counterterm of the form $\phi_1\phi_2$ or $\phi_1\Box\phi_2$ (for general g, λ)? If not, why not?

   (c) Renormalize the theory to one loop order by regularizing (for example with a euclidean momentum cutoff or Pauli Villars), adding the necessary counterterms, and imposing a renormalization condition on the propagators (consider the case where the scalars are both massless) and $2 \to 2$ scattering amplitudes at some values of the kinematical variables s_0, t_0, u_0. Feel free to re-use our results from ϕ^4 theory where appropriate.

(d) Consider the limit of low energies, *i.e.* when $s_0, t_0, u_0 \ll \Lambda^2$ where Λ is the cutoff scale. Tune the location of the poles in both propagators to $p^2 = 0$. Show that the coupling goes to the $\mathsf{O}(2)$-symmetric value if it starts nearby (nearby means $\lambda/g < 3$). (That is, show that at fixed physical coupling, the ratio of bare couplings $\lambda/g \to 1$ as we take the cutoff to infinity.) A nice way to organize this is by computing the beta function for the coupling λ/g.

4. **Yes, please, gauge invariance.** Verify for yourself that the one-loop vacuum polarization amplitude in QED (when computed using either the improved Pauli-Villars regulator or dim reg) satisfies the Ward identity, *i.e.* is proportional to $q^\mu q^\nu - \eta^{\mu\nu} q^2$. It's up to you how much of this to hand in.

5. **Soft gravitons?** [optional] Photons are massless, and this means that the cross sections we measure actually include soft ones that we don't detect. If we don't include them we get IR-divergent nonsense.

   Gravitons are also massless. Do we need to worry about them in the same way? Here we'll sketch some hints for how to think about this question.

   (a) Consider the action

$$S_0[h_{\mu\nu}] = \int d^4x \frac{1}{2} h_{\mu\nu} \Box h^{\mu\nu}.$$

   This is a kinetic term for (too many polarizations of a) two-index symmetric-tensor field $h_{\mu\nu} = h_{\nu\mu}$ (which we'll think of as a small fluctuation of the metric about flat space: $g_{\mu\nu} \simeq \eta_{\mu\nu} + h_{\mu\nu}$, and this is where the coupling below comes from). Like with the photon, we'll rely on the couplings to matter to keep unphysical polarizations from being made. Write the propagator for h. We still raise and lower indices with $\eta_{\mu\nu}$. [1]

[1] A warning: I've done two misdeeds in the statement of this problem. First, the Einstein-Hilbert term is $\int d^4x \frac{1}{8\pi G_N} \sqrt{g} R = \int d^4x \frac{1}{8\pi G_N} (\partial h)^2 + \ldots$ – it has a factor of G_N in front of it. R has units of $\frac{1}{\text{length}^2}$, and g is dimensionless, so G_N has units of length^2 – it is $8\pi G_N = \frac{1}{M_{\text{Pl}}^2}$, where M_{Pl} is the Planck mass. I've absorbed a factor of $\sqrt{G_N}$ into h so that the coefficient of the kinetic term is unity. Second, the $(\partial h)^2$ here involves various index contractions, which I haven't written. Some gauge fixing (de Donder gauge) is required to arrive at the simple expression I wrote above, and one more thing – the $h_{\mu\nu}$ I've written is actually the 'trace-reversed' graviton field

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$$

where $h \equiv \eta^{\mu\nu} h_{\mu\nu}$ is the trace. (I didn't write the bar.) For the details of this, which are not needed for this problem, see chapter 10 of my GR notes.

(b) Couple the graviton to the electron field via

$$S_G = \int d^4x \; G h^{\mu\nu} T_{\mu\nu}$$
$$T_{\mu\nu} \equiv \bar{\psi} \left(\gamma_\mu \partial_\nu + \gamma_\nu \partial_\mu \right) \psi. \tag{3}$$

What are the engineering dimensions of the coupling constant G? What is the new Feynman rule?

(c) Draw a (tree level) Feynman diagram that describes the creation of gravitational radiation from an electron as a result of its acceleration from the absorption of a photon ($e\gamma \to eh$). Evaluate it if you dare. Estimate or calculate the cross section (hint: use dimensional analysis).

(d) Now the main event: study the one-loop diagram by which the graviton corrects the QED vertex. Is it IR divergent? If not, why not?

(e) If you get stuck on the previous part, replace the graviton field by a massless scalar $\pi(x)$. Compare the case of derivative coupling $\lambda \partial_\mu \pi \bar{\psi} \gamma^\mu \psi$ with the more direct Yukawa coupling $y\pi\bar{\psi}\psi$. [Warning: though this example has some similarities with the graviton case, the conclusion is different.]

(f) Quite a bit about the form of the coupling of gravity to matter is determined by the demand of coordinate invariance. This plays a role like gauge invariance in QED. Acting on the small fluctuation, the transformation is

$$h_{\mu\nu}(x) \to h_{\mu\nu}(x) + \partial_\mu \lambda_\nu(x) + \partial_\nu \lambda_\mu(x).$$

What condition does the invariance under this (infinitesimal) transformation impose on the object $T_{\mu\nu}$ appearing in (16).

6. **Equivalent photon approximation.** [optional] Consider a process in which very high-energy electrons scatter off a target. At leading order in α, the electron line is connected to the rest of the diagram by a single photon propagator. If the initial and final energies of the electron are E and E', the photon will carry momentum q with $q^2 = -2EE'(1-\cos\theta)$ (ignoring the electron mass $m \ll E$). In the limit of forward scattering ($\theta \to 0$), we have $q^2 \to 0$, so the photon approaches its mass shell. In this problem, we ask: To what extent can we treat it as a real photon?

(a) The matrix element for the scattering process can be written as

$$\mathcal{M} = -\mathbf{i} e \bar{u}(p') \gamma^\mu u(p) \frac{-\mathbf{i} \eta_{\mu\nu}}{q^2} \hat{\mathcal{M}}^\nu(q)$$

where $\hat{\mathcal{M}}^\nu$ represents the coupling of the virtual photon to the target. Let $q = (q^0, \vec{q})$ and define $\tilde{q} = (q^0, -\vec{q})$. The contribution to the amplitude from the electron line can be parametrized as

$$\bar{u}(p')\gamma^\mu u(p) = Aq^\mu + B\tilde{q}^\mu + C\epsilon_1^\mu + D\epsilon_2^\mu$$

where ϵ_α are unit vectors transverse to $\vec{q}$. Show that B is at most of order θ^2 (dot it with q), so we can ignore it at leading order in an expansion about forward scattering. Why do we not care about the coefficient A?

(b) Working in the frame with $p = (E, 0, 0, E)$, compute

$$\bar{u}(p')\gamma \cdot \epsilon_\alpha u(p)$$

explicitly using massless electrons, where $\bar{u}$ and u are spinors of definite helicity, and $\epsilon_{\alpha=\parallel,\perp}$ are unit vectors parallel and perpendicular to the plane of scattering. Keep only terms through order θ. Note that for $\epsilon_\parallel$, the (small) $\hat{3}$ component matters.

(c) Now write the expression for the electron scattering cross section, in terms of $|\hat{\mathcal{M}}^\mu|^2$ and the integral over phase space of the target. This expression must be integrated over the final electron momentum $\vec{p}'$. The integral over $p^{3'}$ is an integral over the energy loss of the electron. Show that the integral over $p'_\perp$ diverges logarithmically as $p'_\perp$ or $\theta \to 0$.

(d) The divergence as $\theta \to 0$ is regulated by the electron mass (which we've ignored above). Show that reintroducing the electron mass in the expression

$$q^2 = -2(EE' - pp' \cos\theta) + 2m^2$$

cuts off the divergence and gives a factor of $\log(s/m^2)$ in its place.

(e) Assembling all the factors, and assuming that the target cross sections are independent of photon polarization, show that the largest part of the electron-target cross section is given by considering the electron to be the source of a beam of real photons with energy distribution given by

$$N_\gamma(x)dx = \frac{dx}{x}\frac{\alpha}{2\pi}(1 + (1-x)^2)\log\frac{s}{m^2}$$

where $x \equiv E_\gamma/E$. This is the Weiszäcker-Williams equivalent photon approximation. It is a precursor to the theory of jets and partons in QCD.