University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2025
Assignment 7 – Solutions

Due 11:59pm Tuesday, February 25, 2025

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1. **Radiative corrections to Compton scattering.** Check that our prescription for renormalizing QED through one loop (e.g. using Pauli-Villars with renormalization conditions on the electron mass and the coupling $\frac{e^2}{4\pi} = \frac{1}{137}$) suffices to remove all the cutoff dependence in the S matrix for Compton ($e\gamma \to e\gamma$) scattering through $\mathcal{O}(\alpha^2)$.

   [We mostly went over this in lecture, but I did say something slightly wrong at the time.]

   The LSZ formula says

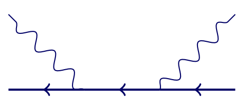
$$S_{e\gamma \leftarrow e\gamma} = \left(\sqrt{Z_e}\right)^2 \left(\sqrt{Z_\gamma}\right)^2 (\text{ amputated diagrams }) \ . \tag{1}$$

   We have a choice about how to do the book-keeping. If we write the photon propagator as

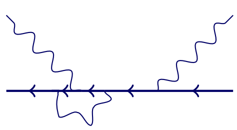
$$\eta_{\mu\nu} \frac{e_0^2 Z_\gamma}{q^2} = \eta_{\mu\nu} \frac{e^2}{q^2} \tag{2}$$

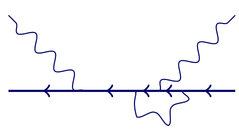
   then there are no factors of Z_γ in the S-matrix. Alternatively, we can say that $Z_\gamma = 1$ and we use the renormalized coupling.

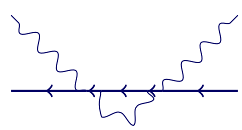
   The amputated diagrams that contribute through one loop are the following, plus the diagram that results from interchanging the two external photon lines, i.e. imposing Bose statistics. At tree level there is:



   The diagrams with divergences are:







   The first diagram replaces the left vertex with

$$\gamma^\mu \rightsquigarrow \gamma^\mu \left(1 + \frac{\alpha}{4\pi} \ln \frac{\Lambda^2}{m^2} + \cdots \right) \ . \tag{3}$$

This cancels the divergence in the

$$\left(\sqrt{Z_e}\right)^2 = 1 - \frac{\alpha}{4\pi} \ln \frac{\Lambda^2}{m^2} + \cdots . \tag{4}$$

(Recall that in $e\mu \to e\mu$ there was only one vertex to correct. Here we have two: The third diagram replaces the right vertex with

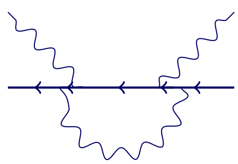
$$\gamma^\nu \rightsquigarrow \gamma^\nu \left(1 + \frac{\alpha}{4\pi} \ln \frac{\Lambda^2}{m^2} + \cdots\right) . \tag{5}$$

This divergence is cancelled by the middle diagram, which replaces the electron propagator with

$$\frac{\mathbf{i}}{\not{p} - m} \rightsquigarrow \frac{\mathbf{i}}{\not{p} - m - \Sigma(\not{p})} \simeq \frac{\mathbf{i} Z_e}{\not{p} - m} = \frac{\mathbf{i}}{\not{p} - m} \left(1 - \frac{\alpha}{4\pi} \ln \frac{\Lambda^2}{m^2} + \cdots\right) \tag{6}$$

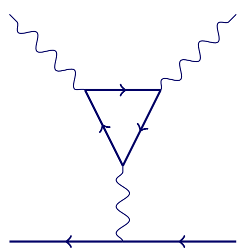
effectively another factor of Z_e, cancelling the singularity in the second vertex correction.

This diagram is actually finite:



$$\sim \int d^4k \frac{1}{\not{k}^3} \frac{1}{k^2}$$

This diagram is zero by Furry's theorem:



I think that's it. This calculation was first done by Brown and Feynman in 1952, https://link.aps.org/doi/10.1103/PhysRev.85.231

2. **Yukawa couplings in QED.** Consider adding to QED an additional scalar field of (physical) mass m, coupled to the electron by

$$L_Y = \lambda \phi \bar{\psi} \psi.$$

Verify that the divergent contribution to the electron wavefunction renormalization factor Z_2 from a virtual ϕ equals the divergent contribution to the QED vertex Z_1 from the one loop correction to the vertex with a virtual ϕ. For an added challenge, verify that the finite parts agree as well.

Since we are only worried about the UV divergences here, in the vertex correction, one only need attend to the ℓ^2 term in the numerator of the integrand. In dim reg, the divergent parts are

$$\delta_1^{div} = 2\lambda^2 \int_0^1 dx \int_0^{1-x} dy \frac{\epsilon - 2}{D} \frac{D}{2} \frac{\Gamma(2 - D/2)}{(4\pi)^{D/2}\Gamma(3)} \bar{\mu}^\epsilon \Delta^{D/2-2}.$$

and

$$\delta_2^{div} = \left(\partial_{\not p} \delta\Sigma|_{\not p = m}\right)^{div} = -\lambda^2 \int_0^1 dx (1-x) \frac{\Gamma(2 - D/2)}{(4\pi)^{D/2}\Gamma(2)} \bar{\mu}^\epsilon \Delta^{D/2-2}$$

Using the identity $\Gamma(1 + x) = x\Gamma(x)$, we have $\frac{D}{2\Gamma(3)} = \frac{1}{\Gamma(2)}$ and $\delta_1^{div} = \delta_2^{div}$ as $D \to 4$.

For purposes of matching the finite parts, some advice: we can put the electron lines on shell and sandwich between spinors satisfying the equations of motion (as we did for the QED vertex correction), and also set the incoming photon momentum $q = p' - p = 0$.

When using dimensional regularization, to get the finite parts to agree it is necessary to continue all appearances of $D = 4$ to D dimensions. In particular, the number of gamma matrices should be D, and in particular one must use the identity:

$$\not\ell \gamma^\mu \not\ell = \frac{\ell^2}{D} \gamma_\nu \gamma^\mu \gamma^\nu = -\frac{D-2}{D} \ell^2 \gamma^\mu.$$

3. **Spectral representation at finite temperature.**

In lecture we have derived a spectral representation for the two-point function of a scalar operator in the vacuum state

$$-\mathbf{i}\mathcal{D}(x) = \langle 0| \mathcal{T}\mathcal{O}(x)\mathcal{O}^\dagger(0) |0\rangle$$

Derive a spectral representation for the two-point function of a scalar operator in thermal equilibrium at a nonzero temperature T:

$$-\mathbf{i}\mathcal{D}_\beta(x) \equiv \mathrm{tr} \frac{e^{-\beta \mathbf{H}}}{Z_\beta} \mathcal{T}\mathcal{O}(x)\mathcal{O}^\dagger(0) = \frac{1}{Z_\beta} \sum_n e^{-\beta E_n} \langle n| \mathcal{T}\mathcal{O}(x)\mathcal{O}^\dagger(0) |n\rangle .$$

Here $Z_\beta \equiv \mathrm{tr} e^{-\beta \mathbf{H}}$ is the thermal partition function. Check that the zero temperature ($\beta \to \infty$) limit reproduces our previous result. Assume that $\mathcal{O} = \mathcal{O}^\dagger$ if you wish.

The idea is again to insert a resolution of the identity in between the operators. All the steps as for the vacuum correlators go through, the only difference being that instead of arriving at a sum of squares of matrix elements of the operator between the vacuum and an arbitrary state, we get matrix elements between pairs of states:

$$\mathbf{i}D(x) = Z_\beta^{-1} \sum_n e^{-\beta E_n} \sum_m \| \mathcal{O}_{nm} \|^2 \left(e^{\mathbf{i}x(p_n - p_m)} \theta(t) + e^{\mathbf{i}x(p_m - p_n)} \theta(-t) \right) .$$

From here, the momentum space representation follows as before. When $\beta \to \infty$, only the groundstate contributes (assume it is nondegenerate) and $Z_\beta \to e^{-\beta E_0}$.

4. **Another consequence of the optical theorem.**

A general statement of the optical theorem is:

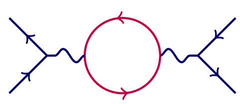
$$-\mathbf{i} \left(\mathcal{M}(a \to b) - \mathcal{M}(b \to a) \right) = \sum_f \int d\Phi_f \mathcal{M}^\star(b \to f) \mathcal{M}(a \to f) .$$

Consider QED with electrons and muons.

(a) Consider scattering of an electron (e^-) and a positron (e^+) into $e^- e^+$ (so $a = b$ in the notation above). We wish to consider the contribution to the imaginary part of the amplitude for this process which is proportional to $Q_e^2 Q_\mu^2$ where Q_e and Q_μ are the electric charges of the electron and muon (which are in fact numerically equal but never mind that). Draw the relevant Feynman diagram, and compute the imaginary part of this amplitude $\text{Im}\,\Pi_\mu(q^2)$ (just the $Q_e^2 Q_\mu^2$ bit) as a function of $s \equiv (k_1 + k_2)^2$ where $k_{1,2}$ are the momenta of the incoming e^+ and e^-. Feel free to re-use results of calculations from lecture.

Check that the imaginary part is independent of the cutoff.

There are a number of diagrams at this order, but the only one that contributes an imaginary part at finite s is the s-channel diagram with a muon loop, that is, where we insert into the photon propagator in the tree level s-channel diagram the contribution to the vacuum polarization from a muon loop (in red):



The key ingredient we've calculated already:

$$\delta\Pi_2(q^2) = \Pi_2(q^2) - \Pi_2(0) = \frac{e^2}{2\pi^2} \int_0^1 dx x(1-x) \log\left(\frac{m^2 - x(1-x)q^2}{m^2} \right) .$$

(Note that without the fermion-loop minus sign, the sign would be opposite.) The imaginary part $\mathrm{Im}\,\delta\Pi_2(q^2+\mathbf{i}\epsilon)$ comes from the locus where the argument of the log is negative (in which case the imaginary part is π), which happens when $m^2 - x(1-x)q^2 < 0$, which happens when $x \in [x_-, x_+] \equiv [\frac{1}{2} - \sqrt{1-m^2/q^2}, \frac{1}{2} + \sqrt{1-m^2/q^2}]$. So

$$\mathrm{Im}\,\delta\Pi_2(q^2) = -\frac{e^2}{2\pi^2}\int_{x_-}^{x_+} dx x(1-x)\pi = -\frac{\alpha}{3}\sqrt{1-4m^2/q^2}\left(1+\frac{2m^2}{q^2}\right).$$

Note that there is also a t-channel diagram proportional to $Q_e^2 Q_\mu^2$, but it does not have an imaginary part.

(b) Use the optical theorem and the fact that the total cross section for $e^+e^- \to \mu^+\mu^-$ must be positive

$$\sigma(e^+e^- \to \mu^+\mu^-) \geq 0$$

to show that a Feynman diagram with a fermion loop must come with a minus sign. Check that with the correct sign, the optical theorem is verified.

Consider forward scattering of e^+e^-, and average over initial spins using

$$\frac{1}{4}\sum_{spins} \bar{u}(k)\gamma^\mu v(k_+)\bar{v}(k_+)\gamma_\mu u(k) = -k\cdot k_+ - 4m_e^2 \simeq -(k+k_+)^2 = -s.$$

(Notice that this is negative!) Recalling that $\Pi_2^{\mu\nu} = q^2\eta^{\mu\nu}\mathbf{i}\Pi_2(q^2)+$ longitudinal terms, gives

$$\mathrm{Im}\,\mathcal{M} = -\frac{s^2}{s^2}\mathrm{Im}\,\Pi_2(q^2) \qquad (7)$$

$$= \frac{e^4}{12\pi}\sqrt{1-\frac{4m^2}{q^2}}\left(1+\frac{2m^2}{q^2}\right) = 2E_{cm}p_{cm}\sigma_{e^+e_-\to L^+L^-} \overset{E\gg m_e}{\simeq} 2s\sigma_{e^+e_-\to L^+L^-}. \qquad (8)$$

If we left out the minus sign, we would get a negative cross section. In fact, this is how Feynman first figured out this particular Feynman rule.