University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2025
Assignment 7

Due 11:59pm Tuesday, February 25, 2025

---

1. **Radiative corrections to Compton scattering.** Check that our prescription for renormalizing QED through one loop (e.g. using Pauli-Villars with renormalization conditions on the electron mass and the coupling $\frac{e^2}{4\pi} = \frac{1}{137}$) suffices to remove all the cutoff dependence in the S matrix for Compton ($e\gamma \to e\gamma$) scattering through $\mathcal{O}(\alpha^2)$.

   [We mostly went over this in lecture, but I did say something slightly wrong at the time.]

2. **Yukawa couplings in QED.** Consider adding to QED an additional scalar field of (physical) mass m, coupled to the electron by

$$L_Y = \lambda\phi\bar{\psi}\psi.$$

   Verify that the divergent contribution to the electron wavefunction renormalization factor Z_2 from a virtual ϕ equals the divergent contribution to the QED vertex Z_1 from the one loop correction to the vertex with a virtual ϕ. For an added challenge, verify that the finite parts agree as well.

3. **Spectral representation at finite temperature.**

   In lecture we have derived a spectral representation for the two-point function of a scalar operator in the vacuum state

$$-\mathbf{i}\mathcal{D}(x) = \langle 0| \mathcal{T}\mathcal{O}(x)\mathcal{O}^\dagger(0) |0\rangle$$

   Derive a spectral representation for the two-point function of a scalar operator in thermal equilibrium at a nonzero temperature T:

$$-\mathbf{i}\mathcal{D}_\beta(x) \equiv \mathrm{tr}\frac{e^{-\beta\mathbf{H}}}{Z_\beta}\mathcal{T}\mathcal{O}(x)\mathcal{O}^\dagger(0) = \frac{1}{Z_\beta}\sum_n e^{-\beta E_n} \langle n| \mathcal{T}\mathcal{O}(x)\mathcal{O}^\dagger(0) |n\rangle .$$

   Here $Z_\beta \equiv \mathrm{tr}e^{-\beta\mathbf{H}}$ is the thermal partition function. Check that the zero temperature ($\beta \to \infty$) limit reproduces our previous result. Assume that $\mathcal{O} = \mathcal{O}^\dagger$ if you wish.

4. **Another consequence of the optical theorem.**

A general statement of the optical theorem is:

$$-\mathbf{i}\left(\mathcal{M}(a \to b) - \mathcal{M}(b \to a)\right) = \sum_f \int d\Phi_f \mathcal{M}^\star(b \to f)\mathcal{M}(a \to f) \ .$$

Consider QED with electrons and muons.

(a) Consider scattering of an electron (e^-) and a positron (e^+) into e^-e^+ (so $a = b$ in the notation above). We wish to consider the contribution to the imaginary part of the amplitude for this process which is proportional to $Q_e^2 Q_\mu^2$ where Q_e and Q_μ are the electric charges of the electron and muon (which are in fact numerically equal but never mind that). Draw the relevant Feynman diagram, and compute the imaginary part of this amplitude $\text{Im}\,\Pi_\mu(q^2)$ (just the $Q_e^2 Q_\mu^2$ bit) as a function of $s \equiv (k_1 + k_2)^2$ where $k_{1,2}$ are the momenta of the incoming e^+ and e^-. Feel free to re-use results of calculations from lecture.

Check that the imaginary part is independent of the cutoff.

(b) Use the optical theorem and the fact that the total cross section for $e^+e^- \to \mu^+\mu^-$ must be positive

$$\sigma(e^+e^- \to \mu^+\mu^-) \geq 0$$

to show that a Feynman diagram with a fermion loop must come with a minus sign. Check that with the correct sign, the optical theorem is verified.