University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2025
Assignment 8

Due 11:59pm Tuesday, March 4, 2025

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1. **Bubble-chain approximation for bound states.**

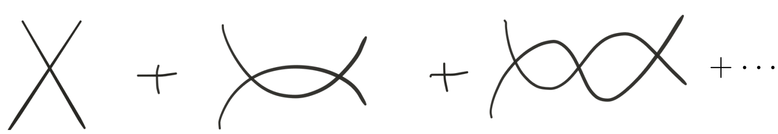
   In discussing the form of the spectral density for an operator that creates a massive particle, I mentioned that in addition to the single-particle delta function at $s = m^2$, and the continuum above $s = (2m)^2$, there could be delta functions associated with bound states at $m^2 < s < (2m)^2$. Here we'll get an idea how we might discover such a thing theoretically.

   For this problem, we're going to work in $D = 2 + 1$ dimensions, so that we can avoid the problem of UV divergences. Consider the theory of a single real scalar with action

   $$S[\phi] = \int d^3x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{g}{4!} \phi^4 \right)$$

   where m, g are real. In this problem we will consider both signs of g, without worrying about questions of the stability of the vacuum (maybe there is a small ϕ^6 term that saves the day but can be ignored here).

   (a) Consider the amplitude $\mathcal{M}(s)$ for elastic scattering $\phi\phi \to \phi\phi$, with $s = E_T^2$, the square of the total center of mass energy. Compute $\mathcal{M}(s)$ in the bubble-chain approximation, defined as the following infinite sum of Feynman diagrams:

   Do not worry about justifying the validity of the approximation (it is not justified in this theory, though it is in a large-n version of the theory), and do not worry about convergence of the series. You can leave your answer as a Feynman parameter integral.

   (b) Show, by explicit calculation, that the bubble chain approximation to the scattering amplitude obeys the optical theorem. [In elastic scattering in the center of mass frame in 3d, the element of solid angle $d\Omega$ is just an element of ordinary angle $d\theta$, and $d\sigma/d\theta = \frac{|\mathcal{M}|^2}{32\pi p E_T^2}$ where p is the magnitude of the spatial momentum of either particle.]

(c) The interaction between the ϕ quanta could result in two of them forming a bound state of mass M_B. A signal of such a bound state is the appearance of a pole in $\mathcal{M}(s)$ at $s = M_B^2$ on the real axis, but below threshold ($0 < M_B^2 < 4m^2$). Find the values of g for which the bubble-chain approximation predicts bound states. [You are not asked to give an analytic expression for M_B.]

2. **Another consequence of unitarity of the S matrix.**

(a) Show that unitarity of S, $S^\dagger S = \mathbb{I} = SS^\dagger$, implies that the transition matrix is *normal*:

$$\mathcal{T}\mathcal{T}^\dagger = \mathcal{T}^\dagger\mathcal{T} \ . \tag{1}$$

(b) What does this mean for the amplitudes $\mathcal{M}_{\alpha\beta}$ (defined as usual by $\mathcal{T}_{\alpha\beta} = \not{\delta}(p_\alpha - p_\beta)\mathcal{M}_{\alpha\beta}$)?

(c) The probability of a transition from α to β is

$$P_{\alpha\to\beta} = |S_{\beta\alpha}|^2 = VT\not{\delta}(p_\alpha - p_\beta)|\mathcal{M}_{\alpha\beta}|^2$$

which is IR divergent. More useful is the transition rate per unit time per unit volume:

$$\Gamma_{\alpha\to\beta} \equiv \frac{P_{\alpha\to\beta}}{VT}.$$

Show that the the total decay rate of the state α is

$$\Gamma_\alpha \equiv \int d\beta \Gamma_{\alpha\to\beta} = 2\mathrm{Im}\,\mathcal{M}_{\alpha\alpha}.$$

(d) Consider an ensemble of states p_α evolving according to the evolution rule

$$\partial_t p_\alpha = -p_\alpha \Gamma_\alpha + \int d\beta p_\beta \Gamma_{\beta\to\alpha}. \tag{2}$$

$S[p] \equiv -\int d\alpha p_\alpha \ln p_\alpha$ is the Shannon entropy of the distribution. Show that

$$\frac{dS}{dt} \geq 0$$

as a consequence of (1). This is a version of the Boltzmann H-theorem.

(e) [Bonus] Notice that we are doing something weird in the previous part by using classical probabilities. This is a special case; more generally, we should describe such an ensemble by a density matrix $\rho_{\alpha\beta}$. Generalize the result of the previous part appropriately.

3. **Abrikosov-Nielsen-Oleson vortex string.**

Consider the Abelian Higgs model in $D = 3 + 1$:

$$\mathcal{L}_h \equiv -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}|D_\mu\phi|^2 - V(|\phi|)$$

where ϕ is a (complex) scalar field of charge q whose covariant derivative is $D_\mu\phi = (\partial_\mu - \mathbf{i}qeA_\mu)\,\phi$, and let's take

$$V(|\phi|) = \frac{\kappa}{2}(|\phi|^2 - v^2)^2$$

for some couplings κ, v. Here we are going to do some interesting classical field theory. Set $q = 1$ for a bit.

(a) Consider a configuration that is independent of x^3, one of the spatial coordinates, and static (independent of time). Show that its energy density (energy per unit length in x^3) is

$$U = \int d^2x\,\left(\frac{1}{2}F_{12}^2 + \frac{1}{2}|D_i\phi|^2 + V(|\phi|)\right).$$

(b) [optional, but used crucially below] Consider the special case where $\kappa = \kappa_0 = \left(\frac{eq}{2}\right)^2$. Assuming that the integrand falls off sufficiently quickly at large $x^{1,2}$, show that

$$U_{\kappa=1} = \int d^2x\,\left(\frac{1}{2}\left(F_{12} + \sqrt{\kappa}\left(|\phi|^2 - v^2\right)\right)^2 + \frac{1}{4}|D_i\phi + \mathbf{i}\epsilon_{ij}D_j\phi|^2 + \sqrt{\kappa}v^2F_{12} - \frac{1}{2}\mathbf{i}\epsilon_{k\ell}\partial_k\left(\phi^\star D_\ell\phi\right)\right)$$

(c) The first two terms in the energy density of the previous part are squares and hence manifestly positive, so setting to zero the things being squared will minimize the energy density. Show that the resulting first-order equations (they are called BPS equations after people with those initials, Bogolmonyi, Prasad, Sommerfeld)[1]

$$0 = (D_i + \mathbf{i}\epsilon_{ij}D_j)\,\phi, \quad F_{12} = -|\phi|^2 + v^2$$

are solved by ($x^1 + \mathbf{i}x^2 \equiv re^{\mathbf{i}\varphi}$)

$$\phi = e^{\mathbf{i}n\varphi}f(r), \quad A_1 + \mathbf{i}A_2 = -\mathbf{i}e^{\mathbf{i}\varphi}\frac{a(r) - n}{r}$$

if

$$f' = \frac{a}{r}f, \;\; a' = r(f^2 - v^2)$$

[1] Let's set $\kappa = 1$ for this discussion; it does not affect the qualitative conclusions.

with boundary conditions

$$a \to 0, f \to v + \mathcal{O}\left(e^{-mr}\right), \quad \text{at } r \to \infty \tag{3}$$

$$a \to n + \mathcal{O}(r^2), f \to r^n(1 + \mathcal{O}(r^2)), \quad \text{at } r \to 0.$$

(For other values of κ, the story is not as simple, but there is a solution with the same qualitative properties. See for example §3.3 of E. Weinberg, *Classical solutions in Quantum Field Theory.*)

(d) The second BPS equation and (3) imply that all the action (in particular the support of F_{12}) is localized near $r = 0$. Evaluate the magnetic flux through the $x^1 - x^2$ plane, $\Phi \equiv \int B \cdot da$ in the vortex configuration labelled by n. Show that the energy density is $U = \frac{v^2}{2} \cdot \Phi$.

(e) Show that the previous result for the flux follows from demanding that the two terms in $D_i\phi$ cancel at large r so that

$$D_i\phi \overset{r\to\infty}{\to} 0 \tag{4}$$

faster than $1/r$. Solve (4) for A_i in terms of ϕ and integrate $\int d^2xF_{12}$.

(f) Argue that a single vortex (string) in the ungauged theory (with global $\mathsf{U}(1)$ symmetry)

$$\mathcal{L} = |\partial\phi|^2 + V(|\phi|)$$

does not have finite energy per unit length. By a vortex, I mean a configuration where $\phi \overset{r\to\infty}{\to} ve^{\mathbf{i}\varphi}$, where $x^1 + \mathbf{i}x^2 = re^{\mathbf{i}\varphi}$.

(g) Consider now the case where the scalar field has charge q. (Recall that in a superconductor made by BCS pairing of electrons, the charged field which condenses has electric charge two.) Show that the magnetic flux in the core of the minimal ($n = 1$) vortex is now (restoring units) $\frac{hc}{qe}$. This is a real thing that people can measure.

4. **BPS conditions from supersymmetry.** [bonus!] What's special about $\kappa = \kappa_0$? For one thing, it is the boundary between type I and type II superconductors (which are distinguished by the size of the vortex core). More sharply, it means the mass of the scalar equals the mass of the vector, at least classically. Moreover, in the presence of some extra fermionic fields, the model with this coupling has an additional symmetry mixing bosons and fermions, namely supersymmetry. This symmetry underlies the special features we found above. Here is an outline (you can do some parts without doing others) of how this works. The logic in part (c) underlies a lot of the progress in string theory since the mid-1990s. Please do not trust my numerical factors.

(a) Add to $\mathcal{L}_h$ a charged fermion Ψ (partner of ϕ) and a neutral Majorana fermion λ (partner of A_μ):

$$\mathcal{L}_f = \frac{1}{2}\mathbf{i}\bar{\Psi}\not{D}\Psi + \mathbf{i}\bar{\lambda}\not{D}\lambda + \bar{\lambda}\Psi\phi + h.c..$$

Consider the transformation rules

$$\delta_\epsilon A_\mu = \mathbf{i}\bar{\epsilon}\gamma_\mu\lambda, \delta_\epsilon\Psi = D_\mu\phi\gamma^\mu\epsilon, \quad \delta_\epsilon\phi = -\mathbf{i}\bar{\epsilon}\Psi, \delta_\epsilon\lambda = -\frac{1}{2}\mathbf{i}\sigma^{\mu\nu}F_{\mu\nu}\epsilon + \mathbf{i}(|\phi|^2 - v)\epsilon$$

where the transformation parameter ϵ is a Majorana spinor (and a grassmann variable). Show that (something like this) is a symmetry of $\mathcal{L} = \mathcal{L}_h + \mathcal{L}_f$. This is $\mathcal{N} = 1$ supersymmetry in $D = 4$.

(b) Show that the conserved charges associated with these transformations Q_α (they are grassmann objects and spinors, since they generate the transformations, via $\delta_\epsilon \text{fields} = [\epsilon_\alpha Q_\alpha + h.c., \text{fields}]$), satisfy the algebra

$$\{Q, \bar{Q}\} = 2\gamma^\mu P_\mu + 2\gamma^\mu\Sigma_\mu \tag{5}$$

where P_μ is the usual generator of spacetime translations and Σ_μ is the *vortex string charge*, which is nonzero in a state with a vortex string stretching in the μ direction. $\bar{Q} \equiv Q^\dagger\gamma^0$ as usual.

(c) If we multiply (5) on the right by γ^0, we get the positive operator $\{Q_\alpha, Q^\dagger_\beta\}$. This operator annihilates states which satisfy $Q\,|BPS\rangle = 0$ for some components of Q. Such a state is therefore invariant under some subgroup of the superymmetry, and is called a BPS state. Now look at the right hand side of (5)$\times\gamma^0$ in a configuration where $\Sigma_3 = \pi n v^2$ and show that its energy density is $E \geq \pi|n|v^2$, with the inequality saturated only for BPS states.

(d) To find BPS configurations, we can simply set to zero the relevant supersymmetry variations of the fields. Since we are going to get rid of the fermion fields anyway, we can set them to zero and consider just the (bosonic) variations of the fermionic fields. Show that this reproduces the BPS equations.