

Physics 215B QFT Winter 2025

Assignment 9 – Solutions

Due 11:59pm Tuesday, March 11, 2025

1. Gauge theory brain-warmers.

Please do 4 of the following 6 problems. The rest are bonus material.

- (a) Show that the *adjoint* representation matrices

$$(T^A)_{BC} \equiv -\mathbf{i}f_{ABC}$$

furnish a $\dim \mathbf{G}$ -dimensional representation of the Lie algebra

$$[T^A, T^B] = \mathbf{i}f_{ABC}T^C \quad .$$

Hint: commutators satisfy the Jacobi identity

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$$

The structure constants f_C^{AB} are part of the definition of the Lie algebra – in *any* representation, the generators satisfy $[T^A, T^B] = \mathbf{i}f_C^{AB}T^C$. This is a property of the algebra, not of any particular representation. The Jacobi identity follows from this fact, by taking the commutator of the BHS with T^D . Reshuffling this identity gives the desired equation (up to a sign which may be flipped by redefining $T \rightarrow -T$).

More abstractly, the operation $B \rightarrow \text{ad}_A(B) \equiv [A, B]$ is called the *adjoint* action of A on B . The Jacobi identity is then the statement that $\text{ad}_A \text{ad}_B(C) - \text{ad}_B \text{ad}_A(C) = \text{ad}_{[A, B]}(C)$, *i.e.* $[\text{ad}_A, \text{ad}_B] = \text{ad}_{[A, B]}$. This is the statement that the map $A \rightarrow \text{ad}_A$ preserves the Lie algebra, and hence gives a representation, which is inevitably called the adjoint representation. In terms of the generators of an arbitrary representation, $\text{ad}_{T^A}T^B = [T^A, T^B] = \mathbf{i}f_{ABC}T^C$, we find an expression for the adjoint generators, which is indeed $(T_{\text{adj}}^A)_{BC} = \mathbf{i}f_{ABC}$ with the opposite sign from what I said.

- (b) Show that if $(T_A)_{ij}$ are generators of a Lie algebra in some unitary representation R , then so are $-(T_A)_{ij}^*$. Convince yourselves that these are the generators of the complex conjugate representation \bar{R} .

We have $[T_A, T_B] = \mathbf{i}f_{ABC}T_C$, so $([T_A, T_B])^* = -\mathbf{i}f_{ABC}T_C^*$ (the structure constants are real for a unitary rep) so $[T_A^*, T_B^*] = -\mathbf{i}f_{ABC}T_C^*$, so $[-T_A^*, -T_B^*] = \mathbf{i}f_{ABC}(-T_C^*)$.

The representation operators in the rep R are $e^{i\alpha^A T^A}$, with α^A real and T^A hermitian (if R is a unitary representation). In the rep \bar{R} , they are $e^{-i\alpha^A (T^A)^*}$, so the generators in this rep are indeed $-(T^A)_{ij}^*$.

- (c) Show that in a basis of Lie algebra generators where $\text{tr} T^A T^B = \lambda \delta^{AB}$, the structure constants f_{ABC} are completely antisymmetric. Start from the Lie algebra $[T^A, T^B] = i f^{ABC} T^C$, multiply the BHS by T^D on the right and take the trace:

$$\lambda i f^{ABD} = \text{tr}[T^A, T^B] T^D = \text{tr} T^A T^B T^D - \text{tr} T^B T^A T^D$$

and now use cyclicity of the trace, to show that this is the same as, *e.g.* $\text{tr}[T^D, T^A] T^B$.

- (d) From the transformation law for A , show that the non-abelian field strength transforms in the adjoint representation of the gauge group.

Mindlessly plugging in, we have

$$\begin{aligned} F_{\mu\nu}^A &\mapsto \partial_\mu (A_\nu^A + \partial_\nu \lambda^A - f_{ABC} \lambda^B A_\nu^C) - (\mu \leftrightarrow \nu) \\ &\quad + f_{ABC} (A_\mu^B + \partial_\mu \lambda^B - f_{BDE} \lambda^D A_\mu^E) (A_\nu^C + \partial_\nu \lambda^C - f_{CFG} \lambda^F A_\nu^G) \\ &= F_{\mu\nu}^A - f_{ABC} \lambda^B \partial_\mu A_\nu^C + f_{ABC} \lambda^B \partial_\nu A_\mu^C - f_{ABC} f_{CFG} \lambda^F A_\mu^B A_\nu^G - f_{ABC} f_{BDE} \lambda^D A_\mu^E A_\nu^C \\ &\quad (1) \\ &= F_{\mu\nu}^A - f_{ABC} \lambda^B \partial_\mu A_\nu^C + f_{ABC} \lambda^B \partial_\nu A_\mu^C - \lambda^D A_\mu^E A_\nu^C (f_{ABC} f_{BDE} + f_{AEB} f_{BDC}) \\ &\quad (2) \\ &= F_{\mu\nu}^A - f_{ABC} \lambda^B \partial_\mu A_\nu^C + f_{ABC} \lambda^B \partial_\nu A_\mu^C - \lambda^D A_\mu^E A_\nu^C f_{ADB} f_{BEC} \\ &\quad (3) \\ &= F_{\mu\nu}^A - f_{ABC} \lambda^B \partial_\mu A_\nu^C + f_{ABC} \lambda^B \partial_\nu A_\mu^C - \lambda^B A_\mu^D A_\nu^E f_{ABC} f_{CDE} \\ &\quad (4) \\ &= F_{\mu\nu}^A - \lambda^B f_{ABC} F_{\mu\nu}^C. \end{aligned} \quad (5)$$

Everywhere we ignored $\mathcal{O}(\lambda^2)$ terms. At step (2) we used the Jacobi identity. At steps (1) and (3) we relabelled dummy indices.

- (e) Show that

$$\text{tr} F \wedge F = d \text{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right).$$

Write out all the indices I've suppressed.

On the LHS, we find $\text{tr} F \wedge F = \text{tr} dA \wedge dA + 2 \text{tr} dA \wedge A \wedge A + \text{tr} A \wedge A \wedge A \wedge A$ (using the fact that two-forms are commutative). The last term vanishes by

cyclicity of the trace:

$$\text{tr} A^4 \equiv \text{tr} A \wedge A \wedge A \wedge A \equiv A^a \wedge A^b \wedge A^c \wedge A^d \text{tr} T^a T^b T^c T^d \quad (6)$$

$$\stackrel{\text{cyclicity}}{=} A^a \wedge A^b \wedge A^c \wedge A^d \text{tr} T^d T^a T^b T^c \quad (7)$$

$$\stackrel{\{A^3, A\}=0}{=} -A^d \wedge A^a \wedge A^b \wedge A^c \text{tr} T^d T^a T^b T^c \quad (8)$$

$$\stackrel{\text{relabel}}{=} -A^a \wedge A^b \wedge A^c \wedge A^d \text{tr} T^a T^b T^c T^d = -\text{tr} A^4. \quad (9)$$

On the RHS we get

$$d \text{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) = \text{tr} dA \wedge dA + A \wedge d^2 A \quad (10)$$

$$+ 2/3 (dA \wedge A \wedge A + A \wedge dA \wedge A + A \wedge A \wedge dA) \quad (11)$$

$$= \text{tr} dA \wedge dA + 2dA \wedge A \wedge A \quad (12)$$

using $d^2 = 0$ and again the fact that a 2-form (such as dA) is commutative.

- (f) [Bonus] If you are feeling under-employed, find ω_{2n-1} such that $\text{tr} F^n = d\omega_{2n-1}$.

2. The field of a magnetic monopole.

We saw that $F = dA$ implies (when A is a smooth, globally well-defined differential form) that $dF = 0$, which means no magnetic charge. If A is singular, dF can be nonzero. Moreover, by a gauge transformation we can move the singularity around and hide it, so that the field is everywhere non-singular.

A magnetic monopole of magnetic charge g is defined by the condition that $\int_{S^2} F = g$, where S^2 is any sphere surrounding the monopole. If the system is spherically symmetric, we can write

$$F = \frac{g}{4\pi} d \cos \theta d\varphi.$$

(In this problem, we'll work on a sphere at fixed distance from the monopole.)

- (a) Show that the vector potential

$$A_N = \frac{g}{4\pi} (\cos \theta - 1) d\varphi$$

gives the correct $F = dA$. Show that it is a well-defined one-form on the sphere except at the south pole $\theta = \pi$.

- (b) Show that the one-form

$$A_S = \frac{g}{4\pi} (\cos \theta + 1) d\varphi$$

also gives the correct $F = dA$. Show that it is well-defined except at the north pole $\theta = 0$.

- (c) Near the equator both $A_{N,S}$ are well-defined. Show that *as long as* $eg \in 2\pi\mathbb{Z}$, these two one-forms differ by a gauge transformation

$$A_S - A_N = \frac{1}{ie} g^{-1}(\theta, \varphi) dg(\theta, \varphi)$$

for $g(\theta, \varphi)$ a $U(1)$ -valued function on the sphere, well-defined away from the poles.

Zee page 249. The required $g(\varphi) = e^{i2\frac{eg}{4\pi}\varphi}$, which is single-valued $g(0) = g(2\pi)$ only under the stated condition (which is Dirac quantization of magnetic charge).

3. Wilson loops in abelian gauge theory at weak and strong coupling.

- (a) At weak coupling, the Wilson loop expectation value is a gaussian integral. In $D = 4$, study the continuum limit of a rectangular loop with time extent $T \gg R$, the spatial extent. Show that this reproduces the Coulomb force. VI.B of [this Kogut review](#) explains this in some detail.

- (b) Consider the weak coupling calculation again for a Wilson loop coupled to a massive vector field. Show that this reproduces an exponentially-decaying force between external static charges.

In this case the propagator is short-ranged, so as long as $R, T \gg m_A^{-1}$ the answer will be $\log \langle W(T, R) \rangle \simeq aR + bT$ a perimeter law.

- (c) [bonus problem] Compute the combinatorial factors in the first few terms of the strong-coupling expansion of the Wilson loop in $U(1)$ lattice gauge theory.

- (d) [bonus problem] Consider the case of lattice gauge theory in two space-time dimensions. In this case, show that the plaquette variables $W(\partial\Box) = \prod_{\ell \in \partial\Box} U_\ell$ are actually independent variables.

In spacetime dimensions larger than two, any 3-volume V gives a relation between the plaquette variables, since

$$\prod_{\Box \in \partial V} U_\Box = 1.$$

This is because $U_{\square} = \prod_{\ell \in \partial \square} U_{\ell}$, and the plaquettes tiling the boundary of V (∂V) have boundaries that precisely cancel out so that the *boundary* of the boundary of V is empty. This is a general deep fact about topology: a boundary has no boundary. (This is the key ingredient in simplicial homology.)

But in $D = 2$, there are no 3-volumes, and hence no relations between the plaquette variables.

Here is another, related deep point: If we didn't realize that the boundaries of the plaquettes making up the boundary of V didn't cancel, we would write, in the abelian case,

$$\prod_{\square \in \partial V} U_{\square} = e^{ie \sum_{\square \in \partial V} \oint_{\partial \square} A} \stackrel{\text{Stokes}}{=} e^{ie \sum_{\square \in \partial V} \int_{\square} F} = e^{ie \int_{\partial V} F} \stackrel{\text{Stokes}}{=} e^{ie \int_V dF}.$$

But this last expression is the number of monopoles n inside the volume V times their charge g . But

$$e^{iegn} = 1$$

is Dirac quantization.

In contrast, in two spacetime dimensions, there *are* no 3-volumes, so the plaquette variables are independent, and we can write the lattice gauge theory path integral, even for a non-abelian group, as

$$Z = \int \prod_{\ell} du_{\ell} e^{-S[U_{\square}]} = \int \prod_{\square} dU_{\square} e^{-S[U_{\square}]}$$

(perhaps up to some overall constant factor). For the special case where S is the Wilson action, the action is linear in the plaquette variables, so

$$Z = \int \prod_{\square} dU_{\square} e^{-\frac{1}{g^2} \sum_{\square} \text{tr} U_{\square}} = \prod_{\square} \left(\int dU_{\square} e^{-\frac{1}{g^2} \text{tr} U_{\square}} \right) = \prod_{\square} z_{\square} = z_{\square}^{\text{Area}}$$

where *Area* denotes the number of plaquettes. The theory just falls apart into independent plaquettes which don't care about each other. This extends to the evaluation of correlators of Wilson loops,

$$\langle W(C) \rangle = \prod_{\square} \left(\frac{\int dU_{\square} U_{\square} e^{-\frac{1}{g^2} \text{tr} U_{\square}}}{z_{\square}} \right) = w(\square)^{\text{Area}(C)}$$

so the area law is exact. There are no propagating degrees of freedom, but the theory is not quite topological – it depends on areas. The object z_{\square} is a combination of characters of the group.