University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2025
Assignment 9

Due 11:59pm Tuesday, March 11, 2025

---

1. **Gauge theory brain-warmers.**

   Please do 4 of the following 6 problems. The rest are bonus material.

   (a) Show that the *adjoint* representation matrices

   $$\left(T^A\right)_{BC} \equiv -\mathbf{i} f_{ABC}$$

   furnish a $\dim \mathsf{G}$-dimensional representation of the Lie algebra

   $$[T^A, T^B] = \mathbf{i} f_{ABC} T^C \quad .$$

   Hint: commutators satisfy the Jacobi identity

   $$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0.$$

   (b) Show that if $(T_A)_{ij}$ are generators of a Lie algebra in some unitary representation R, then so are $-(T_A)^\star_{ij}$. Convince yourselves that these are the generators of the complex conjugate representation $\bar{R}$.

   (c) Show that in a basis of Lie algebra generators where $\mathrm{tr} T^A T^B = \lambda \delta^{AB}$, the structure constants f_{ABC} are completely antisymmetric.

   (d) From the transformation law for A, show that the non-abelian field strength transforms in the adjoint representation of the gauge group.

   (e) Show that

   $$\mathrm{tr} F \wedge F = d \mathrm{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right).$$

   Write out all the indices I've suppressed.

   (f) [Bonus] If you are feeling under-employed, find ω_{2n-1} such that $\mathrm{tr} F^n = d\omega_{2n-1}$.

2. **The field of a magnetic monopole.**

   We saw that $F = dA$ implies (when A is a smooth, globally well-defined differential form) that $dF = 0$, which means no magnetic charge. If A is singular, dF can be nonzero. Moreover, by a gauge transformation we can move the singularity around and hide it, so that the field is everywhere non-singular.

A magnetic monopole of magnetic charge g is defined by the condition that $\int_{S^2} F = g$, where S^2 is any sphere surrounding the monopole. If the system is spherically symmetric, we can write

$$F = \frac{g}{4\pi} d\cos\theta d\varphi.$$

(In this problem, we'll work on a sphere at fixed distance from the monopole.)

(a) Show that the vector potential

$$A_N = \frac{g}{4\pi}(\cos\theta - 1)\, d\varphi$$

gives the correct $F = dA$. Show that it is a well-defined one-form on the sphere except at the south pole $\theta = \pi$.

(b) Show that the one-form

$$A_S = \frac{g}{4\pi}(\cos\theta + 1)\, d\varphi$$

also gives the correct $F = dA$. Show that it is well-defined except at the north pole $\theta = 0$.

(c) Near the equator both $A_{N,S}$ are well-defined. Show that *as long as* $eg \in 2\pi\mathbb{Z}$, these two one-forms differ by a gauge transformation

$$A_S - A_N = \frac{1}{\mathbf{i}e} g^{-1}(\theta, \varphi) dg(\theta, \varphi)$$

for $g(\theta, \varphi)$ a $\mathsf{U}(1)$-valued function on the sphere, well-defined away from the poles.

3. **Wilson loops in abelian gauge theory at weak and strong coupling.**

(a) At weak coupling, the Wilson loop expectation value is a gaussian integral. In $D = 4$, study the continuum limit of a rectangular loop with time extent $T \gg R$, the spatial extent. Show that this reproduces the Coulomb force.

(b) Consider the weak coupling calculation again for a Wilson loop coupled to a massive vector field. Show that this reproduces an exponentially-decaying force between external static charges.

(c) [bonus problem] Compute the combinatorial factors in the first few terms of the strong-coupling expansion of the Wilson loop in $\mathsf{U}(1)$ lattice gauge theory.

(d) [bonus problem] Consider the case of lattice gauge theory in two spacetime dimensions. In this case, show that the plaquette variables $W(\partial\Box) = \prod_{\ell \in \partial\Box} U_\ell$ are actually independent variables.