University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2025
Assignment 10 – Solutions

Due 11:59pm Thursday, March 20, 2025

---

1. **When is the QCD interaction attractive?**

   Write the amplitude for *tree-level* scattering of a quark and antiquark of different flavors (say u and $\bar{d}$) in the t-channel (in Feynman $\xi = 1$ gauge). Compare to the expression for $e\bar{\mu}$ scattering in QED.

   First fix the initial colors of the quarks to be different – say the incoming u is red and the incoming $\bar{d}$ is anti-green. Show that the potential is repulsive.

   Now fix the initial colors to be opposite – say the incoming u is red and the incoming $\bar{d}$ is anti-red – so that they may form a color singlet. Show that the potential is attractive.

   Alternatively or in addition, describe these results in a more gauge invariant way, by characterizing the potential in the color-singlet and color-octet channels.

   You can do this problem either by choosing a specific basis for the generators of $\mathsf{SU}(3)$ in the fundamental (a common one is the Gell-Mann matrices), or using more abstract group theory methods.

   Schwartz p.512.

   The t-channel diagram is identical to the QED amplitude with replacement

   $$e^2 \to g^2 T^a_{3,ij} T^a_{\bar{3},\bar{k}\bar{l}}$$

   where T^a_3 and $T^a_{\bar{3}}$ are the generators of $\mathsf{SU}(3)$ in the fundamental and antifundamental representations, respectively. We saw on a previous homework that these are related by $T_{\bar{3}} = -T^\star_3$.

   A nice way to think about this is: The tensor product of 3 and $\bar{3}$ representations decomposes into irreducible representations as $3 \otimes \bar{3} = 1 \oplus 8$, where the former is the singlet and the latter is the adjoint.

2. **Spinors in other dimensions.** The remaining two problems take place in $2+1$ and $1+1$ dimensions respectively and involve Dirac spinors. The following will be useful.

   (a) Show that for both $D = 2$ or $D = 3$ we can find a *two* dimensional representation of the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. Hint: use the Pauli matrices with appropriate some factors of $\mathbf{i}$.

(b) What is $\mathrm{tr}\gamma^\mu\gamma^\nu$ in these two cases?

(c) What is $\mathrm{tr}\gamma^\mu\gamma^\nu\gamma^\rho$ in $D = 2$ and $D = 3$ respectively?

(d) What is $\mathrm{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$?

3. **Where to find a Chern-Simons term.**

Consider a field theory in $D = 2 + 1$ of a massive Dirac fermion, coupled to a *background* $\mathsf{U}(1)$ gauge field A with action:

$$S[\psi, A] = \int d^3x\bar{\psi}\left(\mathbf{i}\not{D} - m\right)\psi$$

where $D_\mu = \partial_\mu - \mathbf{i}A_\mu$. A_μ is not dynamical for the purposes of this problem. Use the two-component spinors from the previous problem.

(a) Convince yourself that the mass term for the Dirac fermion in $D = 2 + 1$ breaks parity symmetry. By parity symmetry I mean a transformation $\psi(x) \to \Gamma\psi(Ox)$ where $\det O = -1$, and Γ is a matrix acting on the spin indices, chosen so that this operation preserves $\bar{\psi}\not{\partial}\psi$.

First: the definition of parity is an element of $\mathsf{O}(d,1)$ that's not in $\mathsf{SO}(d,1)$, *i.e.* one with $\det(g) = -1$. In three spatial dimensions this is accomplished by $(t, \vec{x}) \to (t, -\vec{x})$. But in two spatial dimensions, this transformation has only two minus signs and so has determinant one – it is just a π rotation. (Certainly $\bar{\psi}\psi$ is invariant under it. And in fact Peskin's argument for the transformation of the Dirac field goes through exactly – it picks up a γ^0.) Instead we must do something like $(t, x, y) \to (t, x, -y)$ (other transformations are related by composing with a rotation).

Now we must figure out what this does to the Dirac spinor. First recall that the clifford algebra in $D = 2 + 1$ can be represented by 2×2 matrices (*e.g.* the Paulis, times some factors of $\mathbf{i}$ to get the squares right) and there is no notion of chirality, since the product of the three Paulis is proportional to the identity. We want an operation on $\psi(t, x, -y)$ which gives back the (massless) Dirac equation:

$$0 = \left(\gamma^0\partial_t + \gamma^1\partial_x + \gamma^2\partial_y\right)\psi(t, x, -y) = \left(\gamma^0\partial_t + \gamma^1\partial_x - \gamma^2\partial_{\tilde{y}}\right)\psi(t, x, \tilde{y})$$

with $\tilde{y} \equiv -y$. Inserting $1 = -\gamma_2^2$ before ψ we have

$$0 = \left(\gamma^0\partial_t + \gamma^1\partial_x - \gamma^2\partial_{\tilde{y}}\right)\left(-\gamma_2^2\right)\psi(t, x, \tilde{y}) = \gamma_2\left(\gamma^0\partial_t + \gamma^1\partial_x + \gamma^2\partial_{\tilde{y}}\right)\gamma_2\psi(t, x, \tilde{y})$$

which is proportional to $\tilde{\not{\partial}}\gamma^2\psi(\tilde{x}) = 0$. We conclude that $P\psi(t, x, y)P = \gamma^2\psi(t, x, -y)$ will work (there is a sign ambiguity in the definition of the transformation).

This gives $\bar{\psi}\psi \mapsto \left(\psi^\dagger\gamma^{2\dagger}\right)\gamma^0\gamma^2\psi = \bar{\psi}\left(\gamma^2\right)^2\psi = -\bar{\psi}\psi$, while $\bar{\psi}\not{D}\psi \to \bar{\psi}\not{D}\psi$. Here we used $(\gamma^\mu)^\dagger\gamma^0 = \gamma^0\gamma^\mu$, and $A_\mu(t,x,y) \to (A_0(t,x,-y), A_x(t,x,-y), -A_y(t,x,-y))_\mu$.

(b) We would like to study the effective action for the gauge field that results from integrating out the fermion field

$$e^{-S_{eff}[A]} = \int [D\psi D\bar{\psi}]e^{-S[\psi,A]}.$$

Focus on the term quadratic in A:

$$S_{eff}[A] = \frac{1}{2}\int đ^D q A_\mu(q)\Pi^{\mu\nu}(q)A_\nu(q) + ...$$

We can compute $\Pi^{\mu\nu}$ by Feynman diagrams. Convince yourself that Π comes from a single loop of ψ with two A insertions.

(c) Evaluate this diagram using dim reg near $D = 3$. Show that, in the low-energy limit $q \ll m$ (where we can't make on-shell fermions),

$$\Pi^{\mu\nu} = a\frac{m}{|m|}\epsilon^{\mu\nu\rho}q_\rho + ...$$

for some constant a. Find a. Convince yourself that in position space this is a Chern-Simons term, $S[A] = \frac{k}{4\pi}\int A\wedge dA$. Find the level k.

The key ingredient is that in $D = 3$ we have $\mathrm{tr}\gamma^\mu\gamma^\nu\gamma^\rho = -2\epsilon^{\mu\nu\rho}$, as you can check for the basis we chose above with the Pauli matrices. Note that this would have been zero in $D = 4$, as in Peskin's calculation on page 247-248. The answer in $D = 2+1$ is then the answer for general D plus this extra term, which also has a factor of m since it comes from expanding out the numerators of the electron propagators:

$$\Pi_2(q)^{\mu\nu} = ... - \frac{\mathbf{i}e^2}{(4\pi)^{D/2}}\int_0^1 dx \frac{\Gamma(2-D/2)}{\Delta^{1/2}}\mathrm{tr}\gamma^\mu\gamma^\nu\gamma^\rho m\left((p+q)_\rho - p_\rho\right) \quad (1)$$

$$= ... + \frac{\mathbf{i}e^2}{4\pi}\frac{m}{|m|}\epsilon^{\mu\nu\rho}q_\nu + ... \quad (2)$$

where the ... is all the terms that are there in other dimensions, plus also the terms from expanding in $m^2 \gg q^2$.

The effective action is then

$$S_{\mathrm{eff}}[A] = \frac{1}{2}\int đ^3 q A_\mu(q)\Pi^{\mu\nu}(q)A_\nu(-q) \quad (3)$$

$$= \frac{e^2}{8\pi^2}\mathrm{sign}(m)\int đ^3 A_\mu(q)A_\nu(-q)\epsilon^{\mu\nu\rho}q_\rho \quad (4)$$

$$= \frac{e^2}{8\pi^2}\mathrm{sign}(m)\int A\wedge dA. \quad (5)$$

Clearly this shows that the mass term is odd under parity, since the Chern-Simons term it generates is proportional to sign(m).

(d) Redo this calculation by doing the Gaussian path integral over ψ.

Roughly:

$$\int [D\psi D\bar{\psi}] e^{-S[\psi,\bar{\psi},A]} = \det\left(\mathbf{i}\not{D} - m\right) = e^{\operatorname{tr}\log\left(\mathbf{i}\not{D}-m\right)}.$$

Therefore

$$S_{\text{eff}}[A] = -\text{Tr }\log\left(\mathbf{i}\not{\partial} - \not{A} - m\right) = -\text{Tr }\log\left(\mathbf{i}\not{\partial} - m\right)\left(1 + \not{A}\left(\mathbf{i}\not{\partial} - m\right)^{-1}\right).$$

The trace Tr is over the space on which $\mathbf{i}\not{D} - m$ acts, which is the space of spinor-valued functions. So it includes the spinor trace tr as well as a sum $\int d^3x$ or $\int đ^d p$. Note that the term linear in A is the familiar tadpole diagram, which vanishes by charge conjugation symmetry or Furry's theorem. We need to expand this in A to second order to get Π, and, using

$$A(\hat{x}) = \int đp e^{-\mathbf{i}p\hat{x}}, \quad f(\mathbf{i}\partial) = \int đq |q\rangle\langle q| f(q)$$

the result is

$$S_{\text{eff}}[A] = \ldots - \frac{1}{2}\int d^3x \,\langle x|\operatorname{tr}\not{A}\left(\mathbf{i}\not{\partial} - m\right)^{-1}\not{A}\left(\mathbf{i}\not{\partial} - m\right)^{-1}|x\rangle \tag{6}$$

$$= \ldots - \frac{1}{2}\int d^3x \int đ^3p_{1,2}\int đ^3q_{1,2} e^{-\mathbf{i}q_1 x}\langle x|p_1\rangle \underbrace{\langle p_1| e^{-\mathbf{i}q_2\hat{x}}|p_2\rangle}_{=\int d^3y e^{-\mathbf{i}q_2 y - \mathbf{i}p_1 y + \mathbf{i}p_2 y} = \not{\delta}^3(q_2 - p_1 + p_2)} \langle p_2|x\rangle$$

$$\operatorname{tr}\left(\not{A}(q_1)\left(\not{p}_1 - m\right)^{-1}\not{A}(q_2)\left(\not{p}_2 - m\right)^{-1}\right) \tag{7}$$

$$= \ldots - \frac{1}{2}\int đ^d q A_\mu(q) A_\nu(-q)\int đ^d p \operatorname{tr}\left(\gamma^\mu \frac{1}{\not{p} - m}\gamma^\nu \frac{1}{\not{p} - \not{q} - m}\right) \tag{8}$$

which is the same as the diagrammatic calculation above.

4. **A bit about the Schwinger model.**

Consider a version of QED in $D = 1+1$:

$$S[A,\psi] = \int d^2x \left(\sum_{\alpha=1}^{N_f}\bar{\psi}_\alpha\left(\mathbf{i}\not{D} - m_\alpha\right)\psi_\alpha - \frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} - \frac{\theta}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu}\right) \tag{9}$$

where each ψ_α is a 2-component spinor, and $\not{D} = \gamma^\mu\left(\partial_\mu + \mathbf{i}A_\mu\right)$. Use the two-component spinors from the problem above.

(a) What are the mass dimensions of the coupling g?

Since A has dimensions of mass, we have $0 = -2[g] - 2 + 4$ which says $[g] = 1$, g is a mass scale. What scale is it? This is the energy scale below which we expect the Coulomb force to confine charge.

The model with $N_f = 1$ was solved by Schwinger in 1962. It is a model of confinement of charge, essentially because, even classically, the Coulomb potential grows linearly in $D = 1 + 1$, just like a confining potential. When $\theta = \pi$, there is a line of first-order phase transitions as a function of m_1, terminating at a critical point at $m_1 = m_{\text{cr}} \approx 0.3335g$. (When $\theta = \pi$ things are less interesting.) We'll set $\theta = \pi$ for the whole problem.

If m_2 is very heavy, the model should reduce the $N_f = 1$ model at very low energies, and so should still have a critical point at some value of m_1. However, we can expect that the critical value of m_1 will be modified by some dependence on m_2 that goes away when $m_2 \to \infty$.

Predict the shape of the critical curve $m_{1\text{cr}}(m_2)$ in the regime $m_2 \gg g$.

If I were feeling mean I would end the question here. Since I am feeling friendly, I will say a few more words. The heavy fermion with mass m_2 will contribute to the vacuum polarization for the gauge field.

(b) Compute the contribution to the vacuum polarization of the gauge field from the heavy fermion field $\Pi^{\mu\nu}(q)$ in the limit $m_2 \gg g$, as a function of m_2 and g, for $q^2 \ll m_2^2$.

The calculation is not so different from the previous problem in that it involves exactly the same diagram, but with $D \to 2$.

$$\mathbf{i}\Pi^{\mu\nu}(q) = -(-\mathbf{i}g)^2\mathbf{i}^2 \int đ^D k \operatorname{tr}\gamma^\mu \frac{\not{k} + m_2}{k^2 - m_2^2}\gamma^\nu \frac{\not{k} + \not{q} + m_2}{(k+q)^2 - m_2^2} \tag{10}$$

$$= -g^2 \int_0^1 dx \int đ^D \ell \frac{N^{\mu\nu}(q)}{(\ell^2 - \Delta)^2} \tag{11}$$

with $\Delta = m_2^2 - x(1-x)q^2$ and $\ell \equiv q + xk$, and the numerator is

$$N^{\mu\nu}(q)/\operatorname{tr}1 = 2\ell^\mu \ell^\nu - \eta^{\mu\nu}\ell^2 - 2x(1-x)q^\mu q^\nu + \eta^{\mu\nu}\left(m^2 + x(1-x)q^2\right) + \text{terms linear in } \ell^\mu \tag{12}$$

just as in the lecture notes section 2.7.1, but adapted to be agnostic about how many spinor components tr1 we have.

It is actually a bit tricky to get out the factor of the transverse projector, which must be there by gauge invariance. If we naively set $D = 2$ too early, we'll find that the coefficient of the ℓ^2 term is $\frac{2}{D} - 1$, and simply vanishes. However, it multiplies a divergence at $D = 2$! The result is then

$$\begin{aligned}
\mathbf{i}\Pi^{\mu\nu}(q) &= -\mathbf{i}g^2 \mathrm{tr}1 \int_0^1 dx \int đ^D \ell_E \frac{\eta^{\mu\nu}\ell_E^2(1-2/D) + \eta^{\mu\nu}(m^2 + x(1-x)q^2) - 2x(1-x)q^\mu q^\nu}{(\ell_E^2 + \Delta)^2} \\
&= -\mathbf{i}g^2 \int_0^1 \mathrm{tr}1 \left(\eta^{\mu\nu} \frac{(1-2/D)\frac{D}{2}}{(4\pi)^{D/2}} \frac{\Gamma(2-D/2-1)}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-D/2-1} \right. && (13) \\
&+ \left. \left(\eta^{\mu\nu}(m^2 + x(1-x)q^2) - 2x(1-x)q^\mu q^\mu\right) \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(2-D/2)}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-D/2} \right) && (14)
\end{aligned}$$

Now $\Gamma(2) = \Gamma(1) = 1$, and $(1-D/2)\Gamma(1-D/2) = \Gamma(2-D/2)$, so we can combine the two terms:

$$\begin{aligned}
\mathbf{i}\Pi^{\mu\nu}(q) &= -\mathbf{i}g^2 \mathrm{tr}1 \int_0^1 dx \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(2-D/2)}{\Delta^{2-D/2}} \left(\eta^{\mu\nu}(-\Delta + m^2 + x(1-x)q^2) - 2x(1-x)q^\mu q^\nu\right) \\
&= \left(\eta^{\mu\nu}q^2 - q^\mu q^\nu\right) \frac{(-\mathbf{i}g^2)2\mathrm{tr}1}{(4\pi)^{D/2}} \int_0^1 dx x(1-x) \frac{\Gamma(2-D/2)}{\Delta^{2-D/2}}. && (15)
\end{aligned}$$

This is the right answer for any D (modulo the CS term in $D = 3$). Setting $D = 2$ and $\mathrm{tr}1 = 2$ for two-component spinors (unlike Peskin), the end result is that $\Pi^{\mu\nu}(q) = (\eta^{\mu\nu}q^2 - q^\mu q^\nu)\,\Pi(q)$ with

$$\Pi(q) = -\frac{g^2}{\pi} \int_0^1 dx \frac{x(1-x)}{\Delta} \overset{q^2 \ll m_2^2}{\approx} -\frac{g^2}{6\pi m_2^2}. \tag{16}$$

(c) Interpret your previous result as a correction to g, i.e. to the coefficient $\frac{1}{g_{\text{eff}}^2}$ of the Maxwell term.

The conclusion is that

$$\frac{1}{g_{\text{eff}}^2} = \frac{1}{g^2}\left(1 + \frac{1}{6\pi}\frac{g^2}{m_2^2}\right). \tag{17}$$

(d) Using the known formula for the critical value of m in the $N_f = 1$ case, predict the critical curve as a function of m_2.

Since the pure $N_f = 1$ model has a critical point at $m_c = 0.33g$, the low-energy theory of the 2-flavor model with one heavy flavor of mass m_2 has a critical point at mass

$$m_{1c} = 0.33 g_{\text{eff}} = 0.33 g \left(1 - \frac{1}{12\pi}\frac{g^2}{m_2^2} + \mathcal{O}\left(\frac{g^4}{m_2^4}\right)\right). \tag{18}$$

I got this problem from a talk I saw today at the KITP. The talk was based on the nice paper 2305.04437, as was my solution.