University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2026
Assignment 1

Due 11:59pm Monday, January 12, 2025

- Please hand in your homework electronically via Canvas. The preferred option is to typeset your homework. It is easy to do and you need to learn to do it anyway as a practicing scientist. I'll provide LaTeX template file with each of the problem sets. If you need help getting set up or have any other questions please email me. I am happy to give TeX advice.
- To hand in your homework, please submit a pdf file through the course's Canvas website, under the assignment labelled hw 01.
- I will post solutions as soon as everyone has submitted their homework.

Thanks in advance for following these guidelines. Please ask me by email if you have any trouble.

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1. **Brain-warmer.** Convince yourself that

$$(\partial_g)^n \, e^{-1/g}|_{g=0} = 0 \;\; \forall n.$$

This means that a function of the form $e^{-1/g}$ does not have a useful series expansion about $g = 0$.

2. **Scale invariant quantum mechanics.**

Consider the action for one quantum variable r with $r > 0$ and

$$S[r] = \int dt \left(\frac{1}{2} m \dot{r}^2 - V(r) \right), \qquad V(r) = \frac{\lambda}{r^2} \; .$$

(a) Show that the (non-relativistic) mass parameter m can be eliminated by a multiplicative redefinition of the field r or of the time t. As a result, convince yourself that the physics of interest here should only depend on the combination $m\lambda$. Show that the coupling $m\lambda$ is dimensionless: $[m\lambda] = 0$.

(b) Show that this action is *scale invariant*, *i.e.* show that the transformation

$$r(t) \to s^{\alpha} \cdot r(st) \tag{1}$$

(for some α which you must determine), (with $s \in \mathbb{R}^+$) is a symmetry.

(c) Find the associated Noether charge $\mathcal{D}$. For this last step, it will be useful to note that the infinitesimal version of (1) is ($s = e^a, a \ll 1$)

$$\delta r(t) = a\left(\alpha + t\frac{d}{dt}\right) r(t).$$

[Hint: In field theory the best way to find the Noether current is the following. If we know that under a transformation $\phi \to \phi_\epsilon$ with parameter ϵ constant in spacetime, the action does not change: $S[\phi] = S[\phi_\epsilon]$ then if we allow $\epsilon = \epsilon(x)$ (infinitesimal) then the variation must be proportional to derivatives of ϵ:

$$\delta S \equiv S[\phi_{\epsilon(x)}] - S[\phi] = \int d^D x \partial_\mu \epsilon j^\mu(x) \tag{2}$$

for some functional of the fields j^μ. The RHS is $\delta S = -\int d^D x \epsilon \partial_\mu j^\mu$ by integration by parts (we assume no boundary). j^μ is the conserved Noether current, which is conserved $\partial_\mu j^\mu = 0$ when evaluated on a solution of the equations of motion. This is because at a solution of the equations of motion, the action is invariant under *any* variation $\delta S = 0$, including $\phi \to \phi_\epsilon$ spacetime-dependent ϵ, which given (2) requires $\partial_\mu j^\mu = 0$. As a result, its time component, integrated over space, is time independent:

$$\frac{d}{dt}Q \equiv \frac{d}{dt}\int d^{D-1}\vec{x} j^0 = -\int \nabla \cdot j = 0$$

(again we ignore boundary terms, in space). This method is superior to formulae you may remember from classical mechanics (like $Q = \frac{\partial L}{\partial \dot{r}}\delta r + F^0$ where $\delta L = \dot{F}^0$) because it makes no assumptions about the dependence of the Lagrangian on $\dot{q}$ and it doesn't require remembering anything.]

(d) Find the position-space Hamiltonian $\mathbf{H}$ governing the dynamics of r. Show that the Schrödinger equation is Bessel's equation

$$\left(-\frac{\partial_r^2}{2m} + \frac{\lambda}{r^2}\right)\psi_E(r) = E\psi_E(r).$$

Show that the Noether charge associated $\mathcal{D}$ with scale transformations ($\equiv$ dilatations) satisfies: $[\mathcal{D}, \mathbf{H}] = -\mathbf{i}\mathbf{H}$. This equation says that the Hamiltonian has a definite scaling dimension, *i.e.* that its scale tranformation is

$\delta \mathbf{H} = \mathbf{i}a[\mathcal{D}, \mathbf{H}] = +a\mathbf{H}$. Note that you should not need to use arcane facts about Bessel functions, only the asymptotic analysis of the equation, in subsequent parts of the problem.

(e) Describe the behavior of the solutions to this equation as $r \to 0$. [Hint: in this limit you can ignore the RHS. Make a power-law ansatz: $\psi(r) \sim r^\Delta$ and find Δ.]

(f) What happens if $2m\lambda < -\frac{1}{4}$? It looks like there is a continuum of negative-energy solutions (boundstates). This is another example of a *too-attractive* potential.

(g) A hermitian operator has orthogonal eigenvectors. We will show next that to make $\mathbf{H}$ hermitian when $2m\lambda < -\frac{1}{4}$, we must impose a constraint on the wavefunctions:

$$\left(\psi_E^\star \partial_r \psi_E - \psi_E \partial_r \psi_E^\star\right)|_{r=\epsilon} = 0 \ , \tag{3}$$

where ϵ is an infinitesimal cutoff. There are two useful perspectives on this condition: one is that the LHS is the probability current passing through the point $r = \epsilon$. If it is nonzero, it means probability is leaking into a hole in the world at $r = \epsilon$.

The other perspective is the following. Consider two eigenfunctions:

$$\mathbf{H}\psi_E = E\psi_E, \quad \mathbf{H}\psi_{E'} = E'\psi_{E'}.$$

Multiply the first equation by $\psi_{E'}^\star$ and integrate; multiply the second by $\psi_E^\star$ and integrate; take the difference (maybe take the complex conjugate of the second term). Show that the result is a boundary term which must vanish when $E = E'$.

(h) Show that the condition (3) is empty for $2m\lambda > -\frac{1}{4}$. In contrast, we must impose a further condition on the eigenfunctions for $2m\lambda < -\frac{1}{4}$ in order to satisfy (3). [Really, we want to impose a boundary condition at $r = 0$, but in order to actually do this, we must impose the condition at $r = \epsilon$ for some UV cutoff ϵ. Notice that this regulator explicitly breaks scale invariance.] Show that the resulting spectrum of boundstates has a *discrete* scale invariance.

[Cultural remark: For some reason I don't know, restricting the Hilbert space in this way is called a *self-adjoint extension*.]

(i) [Extra credit] Consider instead a particle moving in $\mathbb{R}^d$ with a central $1/r^2$ potential, $r^2 \equiv \vec{x} \cdot \vec{x}$,

$$S[\vec{x}] = \int dt \left(\frac{1}{2} m \dot{\vec{x}} \cdot \dot{\vec{x}} - \frac{\lambda}{r^2}\right).$$

Show that the same analysis applies (*e.g.* to the s-wave states) with minor modifications.

[A useful intermediate result is the following representation of (minus) the laplacian in $\mathbb{R}^d$:

$$\vec{p}^2 = -\frac{1}{r^{d-1}}\partial_r \left(r^{d-1}\partial_r\right) + \frac{\hat{L}^2}{r^2}, \quad \hat{L}^2 \equiv \frac{1}{2}\hat{L}_{ij}\hat{L}_{ij}, \quad L_{ij} = -\mathbf{i}\left(x_i\partial_j - x_j\partial_i\right),$$

where $r^2 \equiv x^i x^i$. By 's-wave states' I mean those annihilated by $\hat{L}^2$.]