University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2026
Assignment 2

Due 11:59pm Monday, January 19, 2026

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1. **The vacuum is a fluid with $p = -\rho$.**

   We said in lecture that the vacuum energy density ρ gravitates and that, when positive, its effect is to cause space to inflate – to expand exponentially in time. An important aspect of this phenomenon is that the vacuum fluctuations produce not only an energy density, but a *pressure*, $p = T^i_i$ (no sum on i), of the form $p = -\rho$, which is negative for $\rho > 0$. The vacuum therefore acts as a perfect fluid with $p = -\rho$. (The stress tensor for a perfect fluid in terms of its velocity field u^μ takes the form $T^{\mu\nu} = (p + \rho)u^\mu u^\nu + pg^{\mu\nu}$, so in a frame with $u^\mu = (1, \vec{0}^\mu)$, $T^0_0 = \rho, T^i_i = p$.) Solving Einstein's equations with such a source produces an inflating universe. In this problem we show that this is the case from QFT.

   (a) Show that the energy-momentum tensor for a free relativistic scalar field ($S[\phi] = \int d^D x \sqrt{g}\mathcal{L}, \mathcal{L} = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{m^2}{2}\phi^2$) takes the form

   $$T_{\mu\nu} = a\partial_\mu\phi\partial_\nu\phi - bg_{\mu\nu}\mathcal{L}$$

   with some constants a, b.

   You may do this either by deriving the Noether currents for spacetime translations, or by extracting the response to a variation of the spacetime metric, $T_{\mu\nu}(x) = \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}(x)}$. Here and above $\sqrt{g} \equiv \sqrt{|\det g|}$.

   (b) Reproduce the formal expression for the vacuum energy

   $$\langle 0|\mathbf{H}|0\rangle = V \int đ^d k \frac{1}{2}\hbar\omega_{\vec{k}}$$

   using the two point function

   $$\langle 0|\, \phi(x)^2 \,|0\rangle = \langle 0|\phi(0)\phi(0)|0\rangle = \lim_{\vec{x},t\to 0} \langle 0|\, \phi(x)\phi(0)\, |0\rangle$$

   and its derivatives. (V is the volume of space.)

   (c) Show that the vacuum expectation value of the pressure

   $$\langle 0|T_{ii}|0\rangle$$

   (no sum on i) gives the same answer, up to a sign.

[Hints: You'll find a quite different looking integral from the vacuum energy. Use rotation invariance of the vacuum to simplify the answer. The claim is that however you regulate the integral for the vacuum pressure and $\frac{1}{2}\int đ^d k \omega_k$, you'll get the same answer (as long as the regulator respects the symmetries). A convenient regulator is *dimensional regularization*: simply treat the dimension d as an arbitrary complex number.]

(d) Argue that $p = -\rho$ is required in order that the vacuum energy does not specify a preferred rest frame.

(e) Evaluate the vacuum energy using the Feynman rules. That is, draw this amplitude as a Feynman diagram which is a circle – a line connecting a point to itself – with an operator insertion at the point.

(f) [bonus problem] Show that the resulting vacuum energy momentum tensor ($T_{00} = \rho, T_{ii} = -\rho$ (no sum on i)) is the same as the contribution to the energy-momentum tensor from an action of the form

$$S_{\rm cc} = \int d^D x \sqrt{g} \Lambda$$

where Λ is a constant (the cosmological constant).

If you wish, plug in the FRW ansatz for the metric $ds^2 = -dt^2 + a(t)^2 d\vec{x}^2$ and show that Einstein's equations in the presence of a positive cosmological constant

$$\frac{\delta S[g]}{\delta g_{\mu\nu}(x)} = 0, \text{ with } S[g] = \frac{1}{16\pi G_N}\int d^D x \sqrt{g} R + S_{\rm cc} \tag{1}$$

have the solution $a(t) = e^{Ht}$ for some H determined by Λ and G_N.

2. **Casimir force is regulator-independent.** [Bonus problem] Suppose we use a different regulator for the sum in the vacuum energy $\sum_j \hbar\omega_j$. The regulator we'll use here is an analog of Pauli-Villars. In the notation introduced in the lecture notes, we replace

$$f(d) \rightsquigarrow \frac{1}{2}\sum_{j=1}^{\infty} \omega_j K(\omega_j)$$

where the function K is

$$K(\omega) = \sum_\alpha c_\alpha \frac{\Lambda_\alpha}{\omega + \Lambda_\alpha}.$$

We impose two conditions on the parameters c_α, Λ_α:

- We want the low-frequency answer to be unmodified:

$$K(\omega) \overset{\omega\to 0}{\to} 1$$

– this requires $\sum_\alpha c_\alpha = 1$.

- We want the sum over j to converge; this requires that $K(\omega)$ falls off faster than ω^{-2}. Taylor expanding in the limit $\omega \gg \Lambda_\alpha$, we have

$$K(\omega) \overset{\omega\to\infty}{\to} \frac{1}{\omega}\sum_\alpha c_\alpha \Lambda_\alpha - \frac{1}{\omega^2}\sum_\alpha c_\alpha \Lambda_\alpha^2 + \cdots .$$

  So we also require $\sum_\alpha c_\alpha \Lambda_\alpha = 0$ and $\sum_\alpha c_\alpha \Lambda_\alpha^2 = 0$.

First, verify the previous claims about $K(\omega)$.

Then compute $f(d)$ and show that with these assumptions, the Casimir force is independent of the parameters c_α, Λ_α.

[A hint for doing the sum: use the identity

$$\frac{1}{X} = \int_0^\infty ds e^{-sX}$$

inside the sum to make it a geometric series. To do the remaining integral over s, Taylor expand the integrand in the regime of interest.]

3. **Casimir energy from balls and springs.** [More difficult bonus problem] Regularize the Casimir energy of a 1d scalar field by discretizing space. If you suppose there are $N \equiv d/a \in \mathbb{Z}$ lattice points in the left cavity

$$| \leftarrow d \rightarrow | \longleftarrow \qquad L-d \qquad \longrightarrow |$$

what answer do you find for the force on the middle plate?

[Hint: you will find the wrong answer! The problem is that with these assumptions d cannot vary continuously. One way to allow d to vary continuously (and get the right answer) is to impose $\phi(0) = 0 = \phi(d)$, but do not assume d corresponds to a lattice site.]