University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2026
Assignment 5 – Solutions

Due 11:59pm Monday, February 9, 2026

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1. **Brain-warmer.** [optional]

   Prove the Gordon identities

   $$\bar{u}_2 \left(q^\nu \sigma_{\mu\nu}\right) u_1 = \mathbf{i}\bar{u}_2 \left((p_1+p_2)_\mu - (m_1+m_2)\gamma_\mu\right) u_1$$

   and

   $$\bar{u}_2 \left((p_1+p_2)^\nu \sigma_{\mu\nu}\right) u_1 = \mathbf{i}\bar{u}_2 \left((p_2-p_1)_\mu - (m_2-m_1)\gamma_\mu\right) u_1$$

   where $q \equiv p_2 - p_1$ and $\not{p}_1 u_1 = m_1 u_1, \bar{u}_2 \not{p}_2 = m_2 \bar{u}_2$, using the definitions and the Clifford algebra.

   Notice that the two identities are related by relabeling momenta $p_1 \to -p_1$.

2. **Pauli-Villars practice.**

   Consider a field theory of two scalar fields with

   $$\mathcal{L} = -\frac{1}{2}\phi\Box\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{2}\Phi\Box\Phi - \frac{1}{2}M^2\Phi^2 - g\phi\Phi^2 + \text{counterterms}.$$

   Compute the one-loop contribution to the self-energy of Φ. Use a Pauli-Villars regulator – introduce a second copy of the ϕ field of mass Λ with the wrong-sign propagator.

$$\Sigma_\Phi(p) = \int đ^D k \frac{\mathbf{i}}{k^2-m^2}\frac{\mathbf{i}}{(k+p)^2-M^2}(-\mathbf{i}g)^2 \tag{1}$$
$$= g^2\int_0^1 dx \int đ^D k \frac{1}{\left((1-x)(k^2-m^2)+x((k+p)^2-M^2)\right)^2} \tag{2}$$
$$= g^2\int_0^1 dx \int \frac{đ^D \ell}{(\ell^2-\Delta+\mathbf{i}\epsilon)^2}, \quad \ell = k - px, \Delta = xM^2+(1-x)m^2-p^2x(1-x) \tag{3}$$
$$\equiv g^2 \int_0^1 dx \mathcal{J}(\Delta(m)) \tag{4}$$

The Pauli-Villars regulator replaces the ϕ propagator by

$$\frac{\mathbf{i}}{p^2-m^2} \rightsquigarrow \frac{\mathbf{i}}{p^2-m^2} - \frac{\mathbf{i}}{p^2-\Lambda^2}$$

so that the self energy is replaced by

$$\Sigma_\Phi(p) = g^2 \int_0^1 dx \, (\mathcal{J}(\Delta(m)) - \mathcal{J}(\Delta(\Lambda))) \tag{5}$$
$$= \frac{g^2}{8\pi^2} \int_0^1 dx \log \frac{\Delta(\Lambda^2)}{\Delta(m^2)} \tag{6}$$
$$= \frac{g^2}{8\pi^2} \int_0^1 dx \log \frac{\Delta(\Lambda^2)}{\Delta(m^2)} \tag{7}$$
$$\overset{\Lambda \gg \text{everyone}}{=} \frac{g^2}{8\pi^2} \int_0^1 dx \log \left(\frac{x\Lambda^2}{xM^2 + (1-x)m^2 - p^2 x(1-x)} \right) \tag{8}$$

Actually there is also second diagram, where the fermion emits a single scalar which ends at a fermion bubble:

$$-\mathbf{i}\Sigma_\Phi^{\text{tadpole}}(p) = (-\mathbf{i}g)^2 \int đ^4 k \frac{\mathbf{i}}{k^2 - M^2} \frac{\mathbf{i}}{-m^2}.$$

This is independent of the external momentum, and so only contributes to the mass renormalization. A complication that arises here is that the loop contains only a fermion propagator, so our PV regulator involving only a heavy scalar above will not regularize this divergence. We must also add a heavy fermion ghost field. (Such a step is also required to regulate the corrections to the scalar propagator from a fermion bubble.) I'm going to ignore this diagram below.

Determine the counterterms required to impose that the Φ propagator has a pole at $p^2 = M^2$ with residue 1.

To do this, expand (8) about $p^2 = M^2$:

$$\Sigma_\Phi(p) = \frac{g^2}{8\pi^2} \int_0^1 dx \log \frac{x\Lambda^2}{M^2(1-x)^2 + m^2 x} + (p^2 - M^2) \frac{g^2}{8\pi^2} \int_0^1 dx \frac{x(1-x)}{M^2(1-x)^2 + m^2 x} \tag{9}$$
$$\equiv S_1 + (p^2 - M^2) S_2 \tag{10}$$

and do the x integrals. The mass correction depends on the cutoff like $\log \Lambda$, but δ_Z is independent of the cutoff. The actual expressions (which Mathematica can tell you if you are patient enough) are not very illuminating and I don't want to type them, but it's worth noticing that they are singular when $m = 2M$. Why? When $m = 2M$ the intermediate state can be on shell.

The total contribution to Σ, including the counterterms $\mathcal{L}_{ct} = -\delta Z \frac{1}{2} (\partial\phi)^2 +$

$\delta_{M^2}\frac{1}{2}\Phi^2$ (note my silly sign convention) is

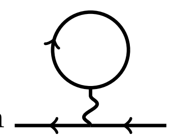
$$0 + (p^2 - M^2)0 + \mathcal{O}(p^2 - M^2)^2 \overset{!}{=} \Sigma(p) - \delta_{M^2} - p^2\delta_Z \tag{11}$$
$$= S_1 + (p^2 - M^2)S_2 - \delta_{M^2} - p^2\delta_Z + \mathcal{O}(p^2 - M^2)^2 \tag{12}$$
$$= S_1 - M^2 S_2 - \delta_{M^2} + p^2(S_2 - \delta_Z) + \mathcal{O}(p^2 - M^2)^2. \tag{13}$$

We conclude that we need to set

$$\delta_M^2 = S_1 - M^2 S_2, \quad \delta_Z = S_2$$

to satisfy the stated renormalization conditions. Notice that in this process, we not only remove the cutoff dependence, but we also determine the *finite* parts of the counterterms.

3. **Tadpole diagrams.**

   (a) Why don't we worry about the following diagram  as a correction to the electron self-energy in QED?

   There are like 12 independent reasons that it is zero. It has to vanish by Lorentz symmetry: the object  would be a source j^μ for the electromagnetic field in the vacuum. At one loop, we can check that $\int đ^4 k \mathrm{tr}\gamma^\mu \frac{\not{k}+m}{k^2-m^2} = 0$ by $\mathrm{tr}\gamma^\mu = 0$ and Lorentz symmetry, $\int^d 4kk^\mu f(k^2) = 0$.

   The one-point function for the photon also has to vanish by charge-conjugation symmetry (in fact any odd-point function of the photon does for the same reason; this is called Furry's theorem).

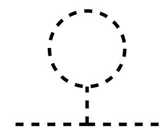
   More generally, a *tadpole diagram* – a diagram with a single field line coming out of it – represents a source for the field. When we developed our Feynman rules, we expanded around a minimum of the potential for the field, and this is why there is no one-point vertex in the Feynman rules. A tadpole diagram is saying that radiative effects are producing a shift in the minimum of the potential. The (quadratic part of the) action wants to change to $\int((\partial A)^2 - m_\gamma^2 A^2 + Aj)$. The equations of motion for the zero-momentum field tell us that the minimum is at $A = j/m_\gamma^2$. In the case of a massless field, the shift is arbitrarily large (in this linear approximation). This is the source of the IR divergence in the tadpole diagram as $m_\gamma \to 0$. In QED, this is moot because $j = 0$.

For the remainder of the problem, we consider ϕ^3 theory with a (small) mass:

$$S = \int d^D x \left(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3 \right).$$

(b) Notice that unlike ϕ^4 theory (or QED), there is no symmetry that forbids a one-point function for the scalar. Why don't we lose generality by not adding a term linear in ϕ to the Lagrangian?

We can shift it away by a field redefinition, $\phi \to \phi - a$. It is convenient to choose a to make the linear term vanish, since then the solution to the equations of motion has $\phi_0 = 0$.

(c) Now think about the following contribution to the scalar self-energy: 

Show that in the limit $m \to 0$ there is an IR divergence. By thinking about the significance for the scalar potential of this part of the diagram  explain the meaning of this divergence.

The object  is a one-point function for the scalar. As explained in the answer to the previous part of the problem, the presence of such a one-point function ($V_{\text{eff}} \ni v\phi$, with $v \propto g$) means we are doing perturbation theory about a configuration which is not a solution to the equations of motion at order g. The correct solution to the equations of motion is ϕ_0 with $0 = m^2\phi_0 + v$ so $\phi_0 = -v/m^2$, which diverges when $m \to 0$. This is the origin of the IR divergence – the field theory is trying to find its minimum which, when $m \to 0$, is arbitrarily far away in field space.

4. **Scale invariance in QFT in $D = 0 + 0$, part 3.** [I got this problem from Frederik Denef.]

We continue our study of QFT in $D = 0 + 0$ with two fields:

$$Z = \int dP_X dP_Y dX dY e^{-H/T}.$$

Let's start by considering again

$$H = \frac{1}{2}P_X^2 + \frac{1}{2}P_Y^2 + V(X,Y), \qquad V(X,Y) = aX^4 + bY^8 \tag{14}$$

for some nonzero constants a, b.

A generic relevant deformation of (14) will flow to a Gaussian fixed point $V(X,Y) \sim X^2+Y^2$ in the IR. Some other, more fine-tuned deformations will flow to other fixed points. For example, $\delta V(X,Y) = \epsilon Y^4$ will flow to $V(X,Y) = X^4+Y^4$. But something more interesting happens for $\delta V(X,Y) = \epsilon X^2Y^2$. This deformation is a relevant perturbation of (14) in the sense that $\delta V(\lambda^{1/4}X, \lambda^{1/8}Y) = \lambda^\kappa V(X,Y)$ with $\kappa = 3/4 < 1$. But it is not true that the model simply flows to a fixed point with $V \propto X^2Y^2$ in the IR. That's because the model with such a potential has a divergent partition function: $\int_{-\infty}^{\infty} dX \int_{-\infty}^{\infty} dY e^{-\epsilon X^2Y^2/T} \propto \sqrt{\frac{T}{\epsilon}} \int \frac{dX}{|X|} = \infty$. We cannot throw away the higher-order terms because they regulate the large-X and large-Y behavior of the integral. Thus, in this model, the UV does not completely decouple from the IR. As a consequence, naive scaling arguments break down, and the partition function develops "anomalous" logarithmic dependence on T for small T.

(a) Compute the partition function for the model (14) deformed by $\delta V(X,Y) = \epsilon X^2Y^2$ analytically using Mathematica or some other symbolic software. This will give a horrible mess of hypergeometric functions. Expand it at small T and you should find something of the form

$$Z = Z_0 T^c \log \frac{\Lambda}{T} \tag{15}$$

up to corrections suppressed by positive powers of $\sqrt{T/\Lambda}$. Find the constants Z_0, c, Λ. Plot the function and compare with your approximate expression in the small-T regime. The overall normalization Z_0 does not mean anything in classical statistical mechanics.

Mathematica will tell you that the integral

$$Z_V = \int_{-\infty}^{\infty} dX dY e^{-(aX^4+bY^8+\epsilon X^2Y^2)/T}$$

is

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Clear[a]; a1 = 4 Integrate[e^(-(a X^4 + b Y^8 + ϵ X^2 Y^2)/T), {X, 0, ∞}, {Y, 0, ∞}, Assumptions → {T > 0, ϵ > 0, a > 0, b > 0}]
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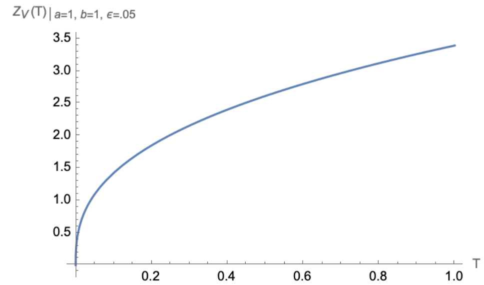
$$\frac{1}{48\, a^{7/4} \sqrt{b^3\, T}} \Bigg(192\, a^{3/2}\, b^{11/8}\, T^{7/8}\, \text{Gamma}\left[\tfrac{9}{8}\right] \text{Gamma}\left[\tfrac{5}{4}\right] \text{HypergeometricPFQ}\left[\left\{\tfrac{1}{8}, \tfrac{1}{8}, \tfrac{5}{8}\right\}, \left\{\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}\right\}, \tfrac{\epsilon^4}{64\, a^2\, b\, T}\right] -$$
$$\epsilon \Bigg(6\, a\, b^{9/8}\, T^{5/8}\, \text{Gamma}\left[\tfrac{3}{8}\right] \text{Gamma}\left[\tfrac{3}{4}\right] \text{HypergeometricPFQ}\left[\left\{\tfrac{3}{8}, \tfrac{3}{8}, \tfrac{7}{8}\right\}, \left\{\tfrac{1}{2}, \tfrac{3}{4}, \tfrac{5}{4}\right\}, \tfrac{\epsilon^4}{64\, a^2\, b\, T}\right] +$$
$$\epsilon \Bigg(-3 \sqrt{a}\, b^{7/8}\, T^{3/8}\, \text{Gamma}\left[\tfrac{5}{8}\right] \text{Gamma}\left[\tfrac{5}{4}\right] \text{HypergeometricPFQ}\left[\left\{\tfrac{5}{8}, \tfrac{5}{8}, \tfrac{9}{8}\right\}, \left\{\tfrac{3}{4}, \tfrac{5}{4}, \tfrac{3}{2}\right\}, \tfrac{\epsilon^4}{64\, a^2\, b\, T}\right] +$$
$$(b^5\, T)^{1/8}\, \epsilon\, \text{Gamma}\left[\tfrac{7}{8}\right] \text{Gamma}\left[\tfrac{7}{4}\right] \text{HypergeometricPFQ}\left[\left\{\tfrac{7}{8}, \tfrac{7}{8}, \tfrac{11}{8}\right\}, \left\{\tfrac{5}{4}, \tfrac{3}{2}, \tfrac{7}{4}\right\}, \tfrac{\epsilon^4}{64\, a^2\, b\, T}\right]\Bigg)\Bigg)\Bigg)$$

This function looks like:



The series expansion has a bit that goes like $\sqrt{T}\log T$ plus corrections of order $\sqrt{T}$, and a bit that goes like $Te^{\frac{\epsilon^4}{64a^2bT}}$. The latter is a very weird function. If it were $e^{-1/T}$ with a negative coefficient in the exponent, it would be easy to say that this is non-perturbatively small. With a positive but small coefficient (*i.e.* for small ϵ) it is essentially indistinguishable from T, as long as $T > 0$. Therefore it is subleading. If you plot each of these bits individually, you can see that the former is the part that matters.

(b) Using (15), compute the dimensionless quantities U/T and C and plot them as a function of T at small T. (Without the logarithmic dependence on T, these would be equal.) Check that in the strict limit $T \to 0$, you get the values for U/T and C that you would have guessed based on naive scaling arguments for $V \propto X^2Y^2$. Note that a logarithm varies more slowly than the $T^{1/2}$ corrections that we threw away.

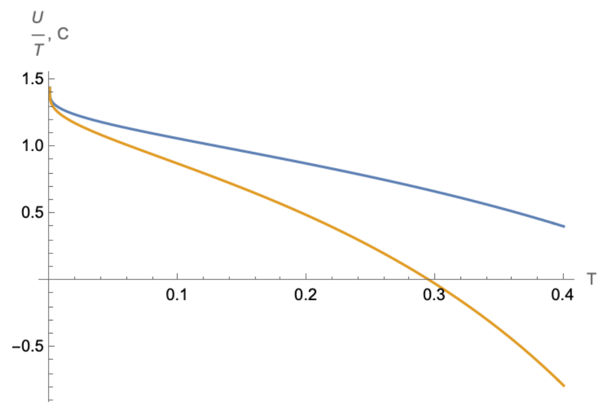
So $Z = Z_0 T^{1+\frac{1}{2}} \log T/\Lambda$ (don't forget the contribution from the two momentum integrals) and therefore

$$U/T = T\partial_T \log Z = \frac{3}{2} + \frac{1}{\log T/\Lambda} \tag{16}$$

while

$$C = \partial_T U = \frac{3}{2} + \frac{1}{\log T/\Lambda} - \frac{1}{\log^2 T/\Lambda}. \tag{17}$$

The naive answer is $Z \sim T^{1+1/2}$, using $Z_V \overset{?}{=} \int dXdY e^{-X^2Y^2\epsilon/T} = \sqrt{T/\epsilon}\int dxdy e^{-x^2y^2}$ by scaling; this would work if the integral were actually well-defined without introducing some other scale. This gives $U/T = C = \frac{3}{2}$, and indeed both of the above functions do approach $\frac{3}{2}$ as $T \to 0$. The correct curves look like



(c) To what extent does the IR physics depend on the UV completion of the $V \propto X^2Y^2$ model? We could have started with $V = aX^8 + bY^8 + \epsilon X^2Y^2$ instead. This model would have different high-temperature physics. Redo part 4a for this potential. You'll find an equally-horrendous, but different combination of hypergeometric functions. Which of the parameters Z_0, c, Λ are the same?

Only c is universal.

(d) The result of the previous part remains true for any other UV completion of the $V \propto X^2Y^2$ model, as long as $\delta V = \epsilon X^2Y^2$ remains a relevant deformation. In fact, we could equally well just take $V = \epsilon X^2Y^2$ and impose a hard cutoff on the X and Y integrals at some fixed values $|X| \leq X_0, |Y| \leq Y_0$ (this is like $V = X^n + Y^n$ with $n \to \infty$). Check that this again reduces to (15).

The answer is simpler:

$$Z_V^L \equiv \int_{-L}^{L} dX \int_{-L}^{L} dY e^{-\epsilon X^2Y^2/T} = 4L^2 \texttt{HypergeometricPFQ}\left[\left\{\frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{3}{2}, \frac{3}{2}\right\}, -\frac{L^4\epsilon}{T}\right].$$

This has the simpler low-temperature expansion:

$$Z_V^L \sim -\sqrt{\frac{\pi T}{\epsilon}} \log \frac{T}{\epsilon L^4 \gamma} + \mathcal{O}(T^{3/2}) + e^{-L^4\epsilon/T}\mathcal{O}(T^2)$$

where γ is some irrelevant constant, and now the other term really is non-perturbatively small.

I just learned (from Rolando Ramirez-Camasca) about the Mathematica command `Asymptotic[]`, which does a better job than `Series[]` here.

(e) In view of this apparent universality of (15) at low T, it is desirable to have a way of deriving it without having to take the detour involving the horrendous hypergeometric functions. Here is one way. We use the hard cutoff $|X| \le L, |Y| \le L$, so that the position-space factor is

$$Z_V(T, L) = \int_{-L}^{L} dX \int_{-L}^{L} dY e^{-X^2Y^2/T} \tag{18}$$

where we've set $\epsilon = 1$ by a choice of temperature units. A rescaling of the integration variables $(X, Y) \to (T^{1/4}X, T^{1/4}Y)$ shows that $Z_V(T, L) = \sqrt{T}F(T^{-1/4}L)$ for some function F of one variable. To find F, compute $L\partial_L Z_V$ directly from (18). By another suitable rescaling, show that $L\partial_L Z$ is finite and easily computable for $L^4/T \to \infty$. Infer from this the dependence on the cutoff L in the regime $T \ll L^4$ and thus the function F in this regime. This reproduces (15).

$$Z_V(T, L) = 4\int_0^L dX \int_0^L dY e^{-X^2Y^2/T} = \sqrt{T}F(T^{-1/4}L).$$

By the fundamental theorem of calculus,

$$L\partial_L Z_V = 4L \int_0^L dY e^{-L^2Y^2/T} \times 2$$

where the last factor of two comes from the place where the L derivative hits the upper limit of the Y integral. By scaling $y = L^2Y^2/T$ (so $dY = dy\sqrt{T}/L$) this is

$$L\partial_L Z_V = 8L\frac{\sqrt{T}}{L}\int_0^{L^2T^{-1/2}} dy e^{-y^2} = 8\sqrt{T}\left(\sqrt{\frac{\pi}{2}} + \mathcal{O}(e^{-L^4/T})\right).$$

Using $x\partial_x|_T = L\partial_L$, we have

$$x\partial x F(x) = T^{-1/2}L\partial_L Z_V = 4\sqrt{\pi} + \mathcal{O}(e^{-L^4/T}).$$

The solution of this ODE is $F(x) = c + 4\sqrt{\pi}\log x$, and therefore

$$Z_V(T, L) = \sqrt{\frac{T}{\epsilon}}(c + \sqrt{\pi}\log\frac{\epsilon L^4}{T}).$$

At the last step, I restored the ϵ by dimensional analysis. Since we don't care about the overall factor, we can get rid of the $\sqrt{\pi}$, and this is what we had above.

(f) We conclude that even when some kind of UV completion is required to give finite answers, the observable low-energy physics remains essentially independent of the UV completion. The infinite number of possible UV completions all flow in the IR to a partition function of the same form (15), with the details of the UV completion all lumped into a single scale parameter Λ. In fact, in the absence of other reference scales that can be used to fix a unit of temperature, the parameter Λ does not really label physically distinct models, since we can always choose units with $\Lambda = 1$. Equivalently, only dimensionless quantities (and relations between them) are physically meaningful. Examples of such dimensionless quantities are C and $u \equiv U/T$. Show that C and u obey a universal relation $C = f(u)$ with $f(u)$ independent of T and Λ, and thus independent of the UV completion of the X^2Y^2 model. In the same spirit, show that the function $g(u)$ in the flow equation $T\partial_T u = g(u)$ is independent of the UV completion.

A brute force way to do this is just to compute them both from $Z = Z_0 T \log T/\Lambda$ and find the answers in (16) and (17). Letting $L \equiv \frac{1}{\log T/\Lambda}$, we have

$$u = \frac{3}{2} + L, C = \frac{3}{2} + L - L^2$$

so $L = u - \frac{3}{2}$ and

$$C = -u^2 + 3u - \frac{3}{2} \equiv f(u).$$

Similarly,

$$T\partial_T u = -\frac{1}{\log T/\Lambda} = -L^2 = -\left(u - \frac{3}{2}\right)^2 \equiv g(u).$$

(g) Show that on the other hand $f(u)$ and $g(u)$ *do* depend on the IR part of the potential, for example by comparing the IR potential $V = X^2Y^2$ considered above to another IR potential such as $V = X^6Y^6$.

If instead we used $\delta V = \epsilon X^6 Y^6$, we would find in part 4e instead

$$Z_V(T, L) = T^{1/6} F(T^{-1/12} L)$$

and

$$x\partial_x F(x) = T^{-\frac{1}{6}} L \partial_L Z_V = 8 \int_0^{L^2 T^{-\frac{1}{6}}} dy e^{-y^6} = 8\Gamma(7/6) + \mathcal{O}(e^{-L^6/\sqrt{T}}).$$

Therefore, in the limit $T \ll L^{12}$, the solution is

$$Z_V = T^{1/6}(c + 8\Gamma(7/6)\log(T^{\frac{-1}{12}} L))$$

and therefore

$$Z = Z_0 T^{1+\frac{1}{6}} \log T/\Lambda$$

and

$$u = U/T = T\partial_T \log Z = \frac{7}{6} + \frac{1}{\log T/\Lambda} = \frac{2}{3} + L \tag{19}$$

while

$$C = \partial_T U = \frac{7}{6} + \frac{1}{\log T/\Lambda} - \frac{1}{\log^2 T/\Lambda} = \frac{7}{6} + L - L^2. \tag{20}$$

These satisfy $L = u - \frac{7}{6}$, so

$$C = u - \left(u - \frac{7}{6}\right)^2 = f(u)$$

and $T\partial_T u = -L^2 = -\left(u - \frac{7}{6}\right)^2 = g(u)$ are indeed different.