University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2026
Assignment 8

Due 11:59pm Monday, March 2, 2025

1. **Brainwarmer: Spectral representation at finite temperature.**

   In lecture we have derived a spectral representation for the two-point function of a scalar operator in the vacuum state

   $$-\mathbf{i}\mathcal{D}(x) = \langle 0| \mathcal{T}\mathcal{O}(x)\mathcal{O}^\dagger(0) |0\rangle$$

   Derive a spectral representation for the two-point function of a scalar operator in thermal equilibrium at a nonzero temperature T:

   $$-\mathbf{i}\mathcal{D}_\beta(x) \equiv \mathrm{tr}\frac{e^{-\beta \mathbf{H}}}{Z_\beta}\mathcal{T}\mathcal{O}(x)\mathcal{O}^\dagger(0) = \frac{1}{Z_\beta}\sum_n e^{-\beta E_n} \langle n| \mathcal{T}\mathcal{O}(x)\mathcal{O}^\dagger(0) |n\rangle \, .$$

   Here $Z_\beta \equiv \mathrm{tr} e^{-\beta \mathbf{H}}$ is the thermal partition function. Check that the zero temperature ($\beta \to \infty$) limit reproduces our previous result. Assume that $\mathcal{O} = \mathcal{O}^\dagger$ if you wish.

2. **Another consequence of the optical theorem.**

   A general statement of the optical theorem is:

   $$-\mathbf{i}\left(\mathcal{M}(a \to b) - \mathcal{M}(b \to a)\right) = \sum_f \int d\Phi_f \mathcal{M}^\star(b \to f)\mathcal{M}(a \to f) \, .$$

   Consider QED with electrons and muons.

   (a) Consider scattering of an electron (e^-) and a positron (e^+) into e^-e^+ (so $a = b$ in the notation above). We wish to consider the contribution to the imaginary part of the amplitude for this process which is proportional to $Q_e^2 Q_\mu^2$ where Q_e and Q_μ are the electric charges of the electron and muon (which are in fact numerically equal but never mind that). Draw the relevant Feynman diagram, and compute the imaginary part of this amplitude $\mathrm{Im}\,\Pi_\mu(q^2)$ (just the $Q_e^2 Q_\mu^2$ bit) as a function of $s \equiv (k_1 + k_2)^2$ where $k_{1,2}$ are the momenta of the incoming e^+ and e^-. Feel free to re-use results of calculations from lecture.

   Check that the imaginary part is independent of the cutoff.

(b) Use the optical theorem and the fact that the total cross section for $e^+e^- \to \mu^+\mu^-$ must be positive

$$\sigma(e^+e^- \to \mu^+\mu^-) \geq 0$$

to show that a Feynman diagram with a fermion loop must come with a minus sign. Check that with the correct sign, the optical theorem is verified.

3. **Bubble-chain approximation for bound states.**

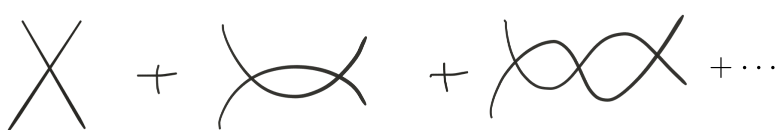
In discussing the form of the spectral density for an operator that creates a massive particle, I mentioned that in addition to the single-particle delta function at $s = m^2$, and the continuum above $s = (2m)^2$, there could be delta functions associated with bound states at $m^2 < s < (2m)^2$. Here we'll get an idea how we might discover such a thing theoretically.

For this problem, we're going to work in $D = 2+1$ dimensions, so that we can avoid the problem of UV divergences. Consider the theory of a single real scalar with action

$$S[\phi] = \int d^3x \left(\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{g}{4!}\phi^4 \right)$$

where m, g are real. In this problem we will consider both signs of g, without worrying about questions of the stability of the vacuum (maybe there is a small ϕ^6 term that saves the day but can be ignored here).

(a) Consider the amplitude $\mathcal{M}(s)$ for elastic scattering $\phi\phi \to \phi\phi$, with $s = E_T^2$, the square of the total center of mass energy. Compute $\mathcal{M}(s)$ in the bubble-chain approximation, defined as the following infinite sum of Feynman diagrams:



Do not worry about justifying the validity of the approximation (it is not justified in this theory, though it is in a large-n version of the theory), and do not worry about convergence of the series. You can leave your answer as a Feynman parameter integral.

(b) Show, by explicit calculation, that the bubble chain approximation to the scattering amplitude obeys the optical theorem. [In elastic scattering in the center of mass frame in 3d, the element of solid angle $d\Omega$ is just an element of ordinary angle $d\theta$, and $d\sigma/d\theta = \frac{|\mathcal{M}|^2}{32\pi p E_T^2}$ where p is the magnitude of the spatial momentum of either particle.]

(c) The interaction between the ϕ quanta could result in two of them forming a bound state of mass M_B. A signal of such a bound state is the appearance of a pole in $\mathcal{M}(s)$ at $s = M_B^2$ on the real axis, but below threshold ($0 < M_B^2 < 4m^2$). Find the values of g for which the bubble-chain approximation predicts bound states. [You are not asked to give an analytic expression for M_B.]