University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2026
Assignment 10 ("Final Exam")

Due 11:59pm Thursday, March 19, 2026

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1. **Brainwarmers on differential forms.** [Optional]

   (a) Prove that for a k-form ω on a d-dimensional manifold with a metric signature having s negative eigenvalues, applying the Hodge star twice yields:

   $$* * \omega = (-1)^{k(d-k)+s}\omega \ .$$

   In the "mostly-minus" convention in 4-dimensional spacetime ($d = 4$, $s = 3$), calculate $* * F$ where F is the electromagnetic field strength 2-form.

   (b) Consider a static point charge q localized at the origin. In spherical coordinates (t, r, θ, ϕ), the mostly-minus flat spacetime metric is:

   $$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \ .$$

   The gauge potential 1-form for this static charge is $A = \frac{q}{4\pi r}dt$. Calculate the electromagnetic field strength 2-form, $F = dA$. Compute the dual field strength 2-form, $*F$. Evaluate the integral of $*F$ over a spatial 2-sphere S^2 of radius R at a constant time t: $\int_{S^2} *F$ Interpret your result physically.

   (c) Let $A = A_\mu dx^\mu$ be the gauge potential 1-form, and $F = dA = \frac{1}{2}F_{\mu\nu}dx^\mu dx^\nu$. Show that the two homogeneous Maxwell equations are equivalent to the statement $dF = 0$ (the Bianchi identity).

   The Maxwell action in the presence of a source current 1-form J is given by:

   $$S = \int_M \left(-\frac{1}{2}F \wedge *F + A \wedge *J\right) \ .$$

   Vary the action with respect to the gauge field $A \to A + \delta A$ to derive the equations of motion (the inhomogeneous Maxwell equations): $d * F = *J$.

   [Hint: the exterior derivative d obeys the generalized Leibniz rule, $d(\alpha\wedge\beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$, where α is a k-form.

   (d) Let's think about the relation between the form notation and the usual electric and magnetic fields in E&M. Write F explicitly in terms of the electric field components E_i and magnetic field components B_i. Compute the dual field strength $*F$. Show that the Lorentz invariant quantity $F \wedge *F$ evaluates to proportional to $(|\mathbf{B}|^2 - |\mathbf{E}|^2) dt\wedge dx\wedge dy\wedge dz$. What is the Lorentz invariant $F \wedge F$ in terms of $\mathbf{E}$ and $\mathbf{B}$?

2. **Gauge theory and Lie algebra brain-warmers.**

Please do 3 of the following 6 problems. The rest are bonus material.

(a) Show that the *adjoint* representation matrices

$$\left(T^A\right)_{BC} \equiv -\mathbf{i} f_{ABC}$$

furnish a dim G-dimensional representation of the Lie algebra

$$[T^A, T^B] = \mathbf{i} f_{ABC} T^C \quad .$$

Hint: commutators satisfy the Jacobi identity

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0.$$

(b) Show that if $(T_A)_{ij}$ are generators of a Lie algebra in some unitary representation R, then so are $-(T_A)^\star_{ij}$. Convince yourself that these are the generators of the complex conjugate representation $\bar{R}$.

(c) Show that in a basis of Lie algebra generators where $\mathrm{tr} T^A T^B = \lambda \delta^{AB}$, the structure constants f_{ABC} are completely antisymmetric.

(d) From the transformation law for A, show that the non-abelian field strength transforms in the adjoint representation of the gauge group.

(e) Show that

$$\mathrm{tr} F \wedge F = d\mathrm{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right).$$

Write out all the indices I've suppressed.

(f) [Bonus] If you are feeling under-employed, find ω_{2n-1} such that $\mathrm{tr} F^n = d\omega_{2n-1}$.

3. **The field of a magnetic monopole.** [Bonus problem]

We saw that $F = dA$ implies (when A is a smooth, globally well-defined differential form) that $dF = 0$, which means no magnetic charge. If A is singular, dF can be nonzero. Moreover, by a gauge transformation we can move the singularity around and hide it, so that the field is everywhere non-singular.

A magnetic monopole of magnetic charge g is defined by the condition that $\int_{S^2} F = g$, where S^2 is any sphere surrounding the monopole. If the system is spherically symmetric, we can write

$$F = \frac{g}{4\pi} d\cos\theta d\varphi.$$

(In this problem, we'll work on a sphere at fixed distance from the monopole.)

(a) Show that the vector potential

$$A_N = \frac{g}{4\pi}\,(\cos\theta - 1)\,d\varphi$$

gives the correct $F = dA$. Show that it is a well-defined one-form on the sphere except at the south pole $\theta = \pi$.

(b) Show that the one-form

$$A_S = \frac{g}{4\pi}\,(\cos\theta + 1)\,d\varphi$$

also gives the correct $F = dA$. Show that it is well-defined except at the north pole $\theta = 0$.

(c) Near the equator both $A_{N,S}$ are well-defined. Show that *as long as* $eg \in 2\pi\mathbb{Z}$, these two one-forms differ by a gauge transformation

$$A_S - A_N = \frac{1}{\mathbf{i}e} g^{-1}(\theta,\varphi)dg(\theta,\varphi)$$

for $g(\theta,\varphi)$ a $\mathsf{U}(1)$-valued function on the sphere, well-defined away from the poles.

4. **When is the QCD interaction attractive?** [Bonus problem]

Write the amplitude for *tree-level* scattering of a quark and antiquark of different flavors (say u and $\bar{d}$) in the t-channel (in Feynman $\xi = 1$ gauge). Compare to the expression for $e\bar{\mu}$ scattering in QED.

First fix the initial colors of the quarks to be different – say the incoming u is red and the incoming $\bar{d}$ is anti-green. Show that the potential is repulsive.

Now fix the initial colors to be opposite – say the incoming u is red and the incoming $\bar{d}$ is anti-red – so that they may form a color singlet. Show that the potential is attractive.

Alternatively or in addition, describe these results in a more gauge invariant way, by characterizing the potential in the color-singlet and color-octet channels.

You can do this problem either by choosing a specific basis for the generators of $\mathsf{SU}(3)$ in the fundamental (a common one is the Gell-Mann matrices), or using more abstract group theory methods.

5. **Spinors in other dimensions.** The remaining two problems take place in $2+1$ and $1+1$ dimensions respectively and involve Dirac spinors. The answers to following questions will be useful.

(a) Show that for both $D = 2$ or $D = 3$ we can find a *two* dimensional representation of the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. Hint: use the Pauli matrices with some appropriate factors of $\mathbf{i}$.

(b) What is $\mathrm{tr}\gamma^\mu\gamma^\nu$ in these two cases?

(c) What is $\mathrm{tr}\gamma^\mu\gamma^\nu\gamma^\rho$ in $D = 2$ and $D = 3$ respectively?

(d) What is $\mathrm{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$?

6. **Where to find a Chern-Simons term.**

Consider a field theory in $D = 2 + 1$ of a massive Dirac fermion, coupled to a *background* $\mathsf{U}(1)$ gauge field A with action:

$$S[\psi, A] = \int d^3x \bar{\psi} \left(\mathbf{i} \not{D} - m \right) \psi$$

where $D_\mu = \partial_\mu - \mathbf{i}A_\mu$. A_μ is not dynamical for the purposes of this problem. Use the two-component spinors from the previous problem.

(a) Convince yourself that the mass term for the Dirac fermion in $D = 2 + 1$ breaks parity symmetry. By parity symmetry I mean a transformation $\psi(x) \to \Gamma\psi(Ox)$ where $\det O = -1$, and Γ is a matrix acting on the spin indices, chosen so that this operation preserves $\bar{\psi}\not{\partial}\psi$.

(b) We would like to study the effective action for the gauge field that results from integrating out the fermion field

$$e^{-S_{eff}[A]} = \int [D\psi D\bar{\psi}] e^{-S[\psi, A]}.$$

Focus on the term quadratic in A:

$$S_{eff}[A] = \frac{1}{2} \int đ^D q A_\mu(q) \Pi^{\mu\nu}(q) A_\nu(q) + ...$$

We can compute $\Pi^{\mu\nu}$ by Feynman diagrams. Convince yourself that Π comes from a single loop of ψ with two A insertions.

(c) Evaluate this diagram using dim reg near $D = 3$. Show that, in the low-energy limit $q \ll m$ (where we can't make on-shell fermions),

$$\Pi^{\mu\nu}(q) = a \frac{m}{|m|} \epsilon^{\mu\nu\rho} q_\rho + ...$$

for some constant a. Find a. Convince yourself that in position space this is a Chern-Simons term, $S[A] = \frac{k}{4\pi} \int A \wedge dA$. Find the level k.

(d) Redo this calculation by doing the Gaussian path integral over ψ.

7. **A bit about the Schwinger model.**

Consider a version of QED in $D = 1+1$:

$$S[A,\psi] = \int d^2x \left(\sum_{\alpha=1}^{N_f} \bar{\psi}_\alpha \left(\mathbf{i} \not{D} - m_\alpha \right) \psi_\alpha - \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} - \frac{\theta}{4\pi} \epsilon_{\mu\nu} F^{\mu\nu} \right) \qquad (1)$$

where each ψ_α is a 2-component spinor, and $\not{D} = \gamma^\mu \left(\partial_\mu + \mathbf{i} A_\mu \right)$. Use the two-component spinors from the problem above.

(a) What are the mass dimensions of the coupling g?

The model with $N_f = 1$ was solved by Schwinger in 1962. It is a model of confinement of charge, essentially because, even classically, the Coulomb potential grows linearly in $D = 1+1$, just like a confining potential. When $\theta = \pi$, there is a line of first-order phase transitions as a function of m_1, terminating at a critical point at $m_1 = m_{\rm cr} \approx 0.3335g$. (When $\theta = \pi$ things are less interesting.) We'll set $\theta = \pi$ for the whole problem.

If m_2 is very heavy, the model should reduce the $N_f = 1$ model at very low energies, and so should still have a critical point at some value of m_1. However, we can expect that the critical value of m_1 will be modified by some dependence on m_2 that goes away when $m_2 \to \infty$.

Predict the shape of the critical curve $m_{1\rm cr}(m_2)$ in the regime $m_2 \gg g$.

If I were feeling mean I would end the question here. Since I am feeling friendly, I will say a few more words. The heavy fermion with mass m_2 will contribute to the vacuum polarization for the gauge field.

(b) Compute the contribution to the vacuum polarization of the gauge field from the heavy fermion field, $\Pi^{\mu\nu}(q)$, in the limit $m_2 \gg g$, as a function of m_2 and g, for $q^2 \ll m_2^2$.

(c) Interpret your previous result as a correction to g, i.e. to the coefficient $\frac{1}{g^2_{\rm eff}}$ of the Maxwell term.

(d) Using the known formula for the critical value of m in the $N_f = 1$ case, predict the critical curve as a function of m_2.

I got this problem from a talk I saw recently at the KITP.